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RESEARCH ARTICLE

Traveling Wave Solutions of Vakhnenko Equation

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ABSTRACT
A new function expansion method is devised for finding traveling wave solutions of nonlinear evolution equation , which can be thought of as the generalization of $(G'/G)-$ *expansion given by M. Wang et al recently. We call it* $(\theta/g)-$ *expansion method. As an application of this new method, we study the well-known Vakhnenko Equation which describes the propagation of high-frequency waves in a relaxing medium. With two new expansions, general types of soliton solutions and periodic solutions for Vakhnenko Equation are obtained.*
KEYWORDS: $(\theta/g)-$ *expansion method, Vakhnenko Equation, Traveling wave solution*
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INTRODUCTION

In the recent decade, the study of nonlinear partial differential equations (NLPDEs) modelling physical phenomena, has become an important tool. In this study, it appears that there are some basic relationships among many complicated nonlinear equations and some simple and solvable nonlinear ordinary differential equations (NODEs) such as Ricatti equation, sine-Gordon equation, sinh-Gordon, Weierstrass elliptic equation etc. In this attempt to use the solutions of NODEs, many powerful approaches have been presented. The investigation of the exact solutions for nonlinear evolution equations plays an important role in the study of soliton theory. In the past decade, a number of powerful methods are proposed, such as the tanh function expansion method [1, 2], Jacobi elliptic function method [3, 4], Exp-function method [5], the hyperbolic tangent function expansion method [6-8], the F-expansion [9-11]. A great number of nonlinear equations can be solved analytically by above methods. [12 - 19] However, although many efforts have been devoted to find various methods to solve nonlinear equation, there is no a unified method. Recently $(G'/G)-$ expansion method [20] has been proposed which can be applied to many nonlinear equations and result in a few new kinds of solution. Then Zhang et al [21] generalized this method to solve nonlinear equations with variable coefficients. Motivated by this method, we introduce the $(\theta/g)-$ expansion which actually is a family of expansion methods. When the θ and g are taken special choice, some familiar expansion methods can be obtained, such as tanh-expansion, $(G'/G)-$ expansion. Based on these interesting results, we further give two new forms of expansion. In order to well illustrate the effectiveness of our method, it is applied to Vakhnenko Equation which is an important equation describing the propagation of high-frequency waves in a relaxing medium. It will be shown that several new types of solution can be derived by using our method.

This paper is organized as follows. Next section is devoted to the description of our method. In Section 3, we apply it to Vakhnenko Equation and discuss briefly its solutions. At Last, a brief summary is given in Section 4.

1. Description of the $(\theta/g)-$ expansion method

A general nonlinear wave equation can be written as following form,

$$P(u,u_t,u_x,u_{tt},u_{xt},u_{xx},...)=0 . \qquad (1)$$

We seek its traveling wave solution $u(\xi)$ by letting

$$\xi = x - Vt\,, \qquad (2)$$

where V is a parameter to be determined later. Now we briefly illustrate the $(\theta/g)-$expansion method.

Step1: Uniting the independent variables x and t into one variable ξ as usual, then Eq. (1) becomes

$$P(u, -Vu', u', V^2u'', -Vu'', u'', \ldots) = 0\,. \qquad (3)$$

Step2: Suppose the solution of equation (3) can be expressed by a polynomial in (θ/g), and θ, g satisfy the following relation:

$$\left(\frac{\theta}{g}\right)' = a + b\left(\frac{\theta}{g}\right) + c\left(\frac{\theta}{g}\right)^2,$$

namely, $$\theta' g - \theta g' = ag^2 + b\theta g + c\theta^2, \qquad (4)$$

where a, b, c are arbitrary constants. Let us examine Eq. (4) carefully. If we take following choice $,\theta = g', a = -\mu, b = -\lambda, c = -1$, then $u(\xi)$ can be expressed as

$$u(\xi) = \sum_{m=0} a_m \left(\frac{g'}{g}\right)^m, \qquad (5)$$

where g satisfies relation $g'' + \lambda g' + \mu g = 0$. It is just the $(G'/G)-$expansion method that M. Wang et al [20] have proposed recently. Furthermore, if we put

$\theta = \tanh\xi$, $g = 1, a = 1, b = 0, c = -1$, and $u(\xi)$ now becomes

$$u(\xi) = \sum_{m=0} a_m (\tanh\xi)^m\,, \qquad (6)$$

which is the $\tanh-$function expansion method.

In the present paper, we propose another two new kinds of expansion from which new solutions of the nonlinear wave equation can be obtained. For the first one, let $\theta = g'/g\,, b = 0$, thus

$$u(\xi) = \sum_{m=0} a_m \left(\frac{g'}{g^2}\right)^m, \qquad (7)$$

where g satisfies

$$g''g^2 - 2gg'^2 = ag^4 + cg'^2\,. \qquad (8)$$

For another, let $\theta = gg'$, then

$$u(\xi) = \sum_{m=0} a_m (g')^m\,. \qquad (9)$$

Now the differential equation about g becomes

$$g'' = a + bg' + cg'^2\,. \qquad (10)$$

Step3: By substituting Eq. (7) or Eq. (9) into Eq. (3), making use of Eq. (8) or Eq. (10), and setting the coefficients of all powers of $(\theta/g)^m$ to zeros, we will get a system of algebraic equations, from which V and a_m can be found explicitly.

Step4: Substituting the values a_m obtained in Step3 back into Eq. (7) or Eq. (9), we may get its all possible solutions.

2. New solutions of Vakhnenko Equation

Vakhnenko Equation [22-24], a nonlinear equation with loop soliton solutions describing the propagation of high-frequency waves in a relaxing medium, can be written as

$$u_{tx} + u_x^2 + uu_{xx} + u = 0\,. \qquad (11)$$

Following Vakhnenko et al [22], we introduce new independent variables X, T, defined by

$$x = T + W(X,T) + x_0,\ t = X, \qquad (12)$$

where $u(x,t) = W_X(X,T)$, x_0 is a arbitrary constant.

From Eq. (12) it follows that

$$\frac{\partial}{\partial X} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x},\ \frac{\partial}{\partial T} = (1+W_T)\frac{\partial}{\partial x}. \qquad (13)$$

Thus Eq. (11) can be rewritten as

$$W_{XXT} + W_X W_T + W_X = 0. \qquad (14)$$

Now we look for the traveling solution of W by putting

$$W = \phi(\xi),\ \xi = X - VT. \qquad (15)$$

Substituting Eq. (15) into Eq. (14), we have

$$-V\phi''' - V(\phi')^2 + \phi' = 0. \qquad (16)$$

Considering the homogeneous balance ϕ''' and $(\phi')^2$ in Eq. (16) and noticing Eq. (4), we require that the highest order of the polynomial in (θ/g) is 1.

3.1 (g'/g^2) − expansion

Suppose

$$\phi(\xi) = a_0 + a_1\left(\frac{g'}{g^2}\right). \qquad (17)$$

By noting $\left(\frac{g'}{g^2}\right)' = a + c\left(\frac{g'}{g^2}\right)^2$, we have the concrete form of ϕ', ϕ'', ϕ''' and $(\phi')^2$, then substitute them into Eq. (16), collect all terms with same order of $\left(\frac{g'}{g^2}\right)$, and set the coefficients of all powers of $\left(\frac{\theta}{g}\right)^m$ to zeros. We will get a system of algebraic equations for a_0, a_1 and V as following,

$$\begin{cases} -2Va_1a^2c - Va_1^2a^2 + a_1a = 0 \\ -8Va_1ac^2 - 2Va_1^2ac + a_1c = 0 \\ -6Va_1c^3 - Va_1^2c^2 = 0 \end{cases}. \qquad (18)$$

After some algebraic calculation, and yields

$$a_1 = -6c,\ V = -\frac{1}{4ac}. \qquad (19)$$

Substituting Eq. (19) and the general solution of Eq. (8) (see Eq. $(A.5)$, Eq. $(A.7)$ and Eq. $(A.9)$ in the Appendix) into Eq. (17), we therefore have two types of solutions as following:

For $ac > 0$,

$$\phi_1(\xi) = a_0 - 6\sqrt{ac}\,\frac{C_1\cos\left(\sqrt{ac}\xi\right) + C_2\sin\left(\sqrt{ac}\xi\right)}{C_1\sin\left(\sqrt{ac}\xi\right) - C_2\cos\left(\sqrt{ac}\xi\right)}, \qquad (20)$$

where $\xi = X + \frac{1}{4ac}T$ and C_1, C_2 are arbitrary constants.

Thus the solution of Vakhnenko Equation is

$$u_1(x,t)=\frac{6ac\left(C_1^2+C_2^2\right)}{\left(C_1\sin\left(\sqrt{ac}\xi\right)-C_2\cos\left(\sqrt{ac}\xi\right)\right)^2},$$

$$x+4act=4ac\xi+x_0+a_0-6\sqrt{ac}\,\frac{C_1\cos\left(\sqrt{ac}\xi\right)+C_2\sin\left(\sqrt{ac}\xi\right)}{C_1\sin\left(\sqrt{ac}\xi\right)-C_2\cos\left(\sqrt{ac}\xi\right)}. \qquad (21)$$

u_1 here had not been given in Ref.[22-24]. It is a general form of periodic solution.

For $ac<0$,

$$\phi_2(\xi)=a_0+6\sqrt{|ac|}\left(\frac{C_1e^{2\xi\sqrt{|ac|}}+C_2}{C_1e^{2\xi\sqrt{|ac|}}-C_2}\right), \qquad (22)$$

where $\xi=X+\frac{1}{4ac}T$ and C_1,C_2 are arbitrary constants. In particular, if C_1 and C_2 take the special value, for example, $C_2/C_1=-e^{2\xi_0}$, then

$\phi_2(\xi)=a_0+6\sqrt{|ac|}\tanh\left(\sqrt{|ac|}\xi+\xi_0\right)=a_0+\frac{3}{\sqrt{V}}\tanh\left(\frac{\xi}{2\sqrt{V}}+\xi_0\right)$, which has same form as Ref.[22].

In general, the soliton solution is

$$u_2(x,t)=-6ac+6ac\left(\frac{C_1e^{2\xi\sqrt{|ac|}}+C_2}{C_1e^{2\xi\sqrt{|ac|}}-C_2}\right)^2,$$

$$x+4act=4ac\xi+x_0+a_0+6\sqrt{|ac|}\left(\frac{C_1e^{2\xi\sqrt{|ac|}}+C_2}{C_1e^{2\xi\sqrt{|ac|}}-C_2}\right). \qquad (23)$$

3.2 g' − expansion

Let

$$\phi(\xi)=b_0+b_1g'. \qquad (24)$$

Similarly, noting $(g')'=a+bg'+cg'^2$, one substitutes the new form of ϕ', ϕ'', ϕ''' and $(\phi')^2$ into Eq. (16), and gets

$$\begin{cases} -V\left(ab^2b_1+2a^2b_1c\right)-Vb_1^2a^2+b_1a=0 \\ -V\left(b_1b^3+8b_1abc\right)-2Vb_1^2ab+b_1b=0 \\ -V\left(7b_1b^2c+8b_1ac^2\right)-V\left(b_1^2b^2+2b_1^2ac\right)+b_1c=0 \\ -12Vb_1bc^2-2Vb_1^2bc=0 \\ -6Vb_1c^3-Vb_1^2c^2=0 \end{cases}. \qquad (25)$$

Then we have $b_1=-6c$, $V=-\frac{1}{4ac-b^2}$. (26)

Substituting Eq. (26) and the general solution of Eq. (10) (see Eq. $(A.18)$-$(A.20)$ in the Appendix) into Eq. (24), we have three types of traveling wave solutions of the Vakhnenko Equation as follows:

Case 1: When $\Delta\equiv 4ac-b^2<0$,

$$\phi_3(\xi)=b_0-3\left[\sqrt{-\Delta}\tanh\left(-\frac{\sqrt{-\Delta}}{2}\xi\right)-b\right]$$

$$=\left(b_0+3b\right)+\frac{3}{\sqrt{V}}\tanh\left(\frac{\xi}{2\sqrt{V}}\right), \qquad (27)$$

Therefore, we have

$$u_3\left(x,t\right)=-\frac{3\Delta}{2}\operatorname{sech}^2\left(-\frac{\sqrt{-\Delta}}{2}\xi\right),$$

$$x+\Delta t=\Delta\xi+b_0-3\left[\sqrt{-\Delta}\tanh\left(-\frac{\sqrt{-\Delta}}{2}\xi\right)-b\right]+x_0. \qquad (28)$$

which is the one-loop soliton solution[22-25].

Case 2: When $\Delta>0$,

$$\phi_4(\xi)=b_0-3\left(\sqrt{\Delta}\tan\left(\frac{\sqrt{\Delta}}{2}\xi\right)-b\right), \qquad (29)$$

$$u_4\left(x,t\right)=-\frac{3\Delta}{2}\sec^2\left(\frac{\sqrt{\Delta}}{2}\xi\right),$$

$$x+\Delta t=\Delta\xi+b_0-3\left(\sqrt{\Delta}\tan\left(\frac{\sqrt{\Delta}}{2}\xi\right)-b\right)+x_0. \qquad (30)$$

u_4 here had not been given in Ref.[22-25]. Obviously, u_3 and u_4 are the special case of u_1 and u_4. Compared with $g'-$expansion, the $\left(g'/g^2\right)-$expansion is a more powerful tool to explore the solutions for nonlinear evolution equations.

SUMMARY

In this work, the $\left(\theta/g\right)-$expansion method has been proposed which is the generalization of $\left(G'/G\right)-$expansion method. With two new expansions, several types of traveling solutions of the Vakhnenko Equation are obtained, such as periodic solution and loop soliton solution. As far as we know, some solutions are first found. It is also proved that $\left(g'/g^2\right)-$expansion is more effective than the $g'-$expansion because the former can give a general form of periodic solutions and soliton solutions while the latter can not. Though this new method only represents the unification of several expansion methods, we believe it may contribute to finding a method that can solve most of nonlinear equation and obtain many new types of solution.

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Appendix：

In this section, general solutions of Eq. (2.8) and (2.10) will be given.

（1） $g''g^2-2gg'^2=ag^4+cg'^2$

Let $g=1/y$, then above equation becomes

$$y''+cy'^2+a=0, \qquad (A.1)$$

which has a general solutions as following,

when $ac>0$, $$y(\xi)=\frac{1}{2c}\ln\left[\frac{c}{a}\left(C_1\sin\left(\sqrt{ac}\xi\right)-C_2\cos\left(\sqrt{ac}\xi\right)\right)^2\right] \qquad (A.2)$$

when $ac<0$, $$y(\xi)=-\frac{1}{2c}\left(2\sqrt{|ac|}\xi-\ln\left[\frac{c}{4a}\left(C_1e^{2\xi\sqrt{|ac|}}-C_2\right)^2\right]\right) \qquad (A.3)$$

when $a=0, c\neq 0$, $$y(\xi)=\frac{1}{c}\ln\left(C_1\xi c+C_2c\right) \qquad (A.4)$$

Thus, we have

when $ac>0$, $$g(\xi)=\frac{2c}{\ln\left[\frac{c}{a}\left(C_1\sin\left(\sqrt{ac}\xi\right)-C_2\cos\left(\sqrt{ac}\xi\right)\right)^2\right]} \qquad (A.5)$$

$$\frac{g'}{g^2}=\sqrt{\frac{a}{c}}\frac{C_1\cos\left(\sqrt{ac}\xi\right)+C_2\sin\left(\sqrt{ac}\xi\right)}{C_1\sin\left(\sqrt{ac}\xi\right)-C_2\cos\left(\sqrt{ac}\xi\right)} \qquad (A.6)$$

when $ac<0$, $$g(\xi)=-\frac{2c}{2\sqrt{|ac|}\xi-\ln\left[\frac{c}{4a}\left(C_1e^{2\xi\sqrt{|ac|}}-C_2\right)^2\right]} \qquad (A.7)$$

$$\frac{g'}{g^2}=\frac{1}{2c}\left(2\sqrt{|ac|}-\frac{4\sqrt{|ac|}C_1e^{2\xi\sqrt{|ac|}}}{C_1e^{2\xi\sqrt{|ac|}}-C_2}\right) \qquad (A.8)$$

when $a=0, c\neq 0$, $$g(\xi)=\frac{c}{\ln\left(C_1\xi c+C_2c\right)} \qquad (A.9)$$

$$\frac{g'}{g^2}=-\frac{C_1}{C_1\xi c+C_2c} \qquad (A.10)$$

（2） $g''=a+bg'+cg'^2$

By putting $y=g'$, $\Delta=4ac-b^2$, the above equation becomes

$$y'=a+by+cy^2$$

or $$\frac{dy}{a+by+cy^2}=d\xi \qquad (A.11)$$

Integrating both sides of $(A.11)$, one has

$$\xi = \int \frac{dy}{a+by+cy^2} = \frac{1}{\sqrt{-\Delta}} \ln \frac{b+2cy-\sqrt{-\Delta}}{b+2cy+\sqrt{-\Delta}} = \frac{-2}{\sqrt{-\Delta}} Arcth \frac{b+2cy}{\sqrt{-\Delta}} \quad \text{for } \Delta < 0 \quad (A.12)$$

$$= \frac{-2}{b+2cy} \quad \text{for } \Delta = 0 \quad (A.13)$$

$$= \frac{2}{\sqrt{\Delta}} \arctan \frac{b+2cy}{\sqrt{\Delta}} \quad \text{for } \Delta > 0 \quad (A.14)$$

, where we have set the integration constant to zero.
Therefore,

$$y = \frac{1}{2c}\left[\sqrt{-\Delta}\tanh\left(-\frac{\sqrt{-\Delta}}{2}\xi \right) - b \right], \quad \text{for } \Delta < 0 \quad (A.15)$$

$$y = -\frac{1}{c\xi} - \frac{b}{2c}, \quad \text{for } \Delta = 0 \quad (A.16)$$

$$y = \frac{1}{2c}\left[\sqrt{\Delta}\tan\left(\frac{\sqrt{\Delta}}{2}\xi \right) - b \right], \quad \text{for } \Delta > 0 \quad (A.17)$$

$$g = \frac{1}{2c}\left[\ln\left(\tanh^2\left(\frac{\sqrt{-\Delta}}{2}\xi \right) - 1 \right) - b\xi \right], \quad \text{for } \Delta < 0 \quad (A.18)$$

$$g = -\frac{1}{c}\left[\ln(\xi) + \frac{b}{2}\xi \right], \quad \text{for } \Delta = 0 \quad (A.19)$$

$$g = \frac{1}{2c}\left[\ln\left(1 + \tan^2\left(\frac{\sqrt{\Delta}}{2}\xi \right) \right) - b\xi \right], \quad \text{for } \Delta > 0 \quad (A.20)$$