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RESEARCH ARTICLE

Exact Traveling Wave Solutions of the Bogoyavlenskii Equation

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ABSTRACT

In this work, exact solutions for the Bogoyavlenskii equation are studied by the $\left(\frac{G'}{G}\right)$*-expansion method. The solutions obtained by this method are general solutions and a variety of special solutions like periodic, kink, antikink,bell type solitons etc. can easily be derived from the general results under certain domain. The work confirms the significant features of the employed method and shows the variety of the obtained solutions. The method is straightforward and concise, and its application are promising.*
KEY WORDS: *the Bogoyavlenskii equation, Solitons, periodic solutions.*
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INTRODUCTION

Nonlinear evolution equations are frequently used to describe many problems of solid state physics, fluid mechanics, plasma physics, population dynamics, chemical kinetics, nonlinear optics, protein chemistry, theory of Bose-Einstein condensates etc.[1]. The basic strategies one may adopt to predict, control and quantify the underlying features of a system under investigation is to model the system in terms of mathematical equations, which are generally nonlinear and then find exact analytic solutions of such model equations using some suitable methods.

In the last few decades, considerable efforts have been made to obtain exact analytical solutions of such nonlinear equations and a number of powerful and efficient methods have been developed for obtaining explicit traveling wave solutions[1, 2, 3, 4, 5, 6].

Very recently, a new powerful technique called $\left(\frac{G'}{G}\right)$-expansion method [7] was introduced for a reliable treatment of nonlinear wave equations. Thereafter some more applications of this method have also been reported [8, 9, 10]. A simplified version of $\left(\frac{G'}{G}\right)$-expansion method is also reported recently [11]. Recently we also exploited this method and obtained some interesting results of a number of equations of physical relevance [10]. With a motivation to further expand the domain of applications of $\left(\frac{G'}{G}\right)$-expansion method, here in the present work, we study of exact solutions for the Bogoyavlenskii equation [12].

$$\begin{aligned} &4u_t + u_{xxy} - 4u^2 u_y - 4u_x v = 0, \\ &u u_y = v_x. \end{aligned} \tag{1}$$

In ref. [12], the Lax pair and a nonisospectral condition for the spectral parameter is presented. Eq. (1) was again derived by Kudryashov and Pickerling [13] as a member of a (2+1) Schwarzian breaking soliton hierarchy, and rational solutions of it were obtained. The equation also appeared in [14] as one of the equations associated to nonisospectral scattering problems. The Painleve property of eq.(1) is recently checked by Estevez et al [15]. Some exact solutions of this equation are also found in [16].

Eq. (1), as the modified version of a breaking soliton equation, $4u_{xt}+8u_x u_{xy}+4u_y\ u_{xx}+u_{xxxy}=0$, *describes* the (2+1)-dimensional interaction of a Riemann wave propagating along the y-axis with a long wave along the x-axis. It is well-known that the solution and its dynamics of the equation can make researchers deeply understand the described physical process.

The organization of the paper is as follows: In section 2, a brief account of the $\left(\frac{G'}{G}\right)$-expansion method for finding the traveling wave solutions of nonlinear equations is given. In sections 3, exact solutions of the Bogoyavlenskii equation by this method are given. Finally concluding remarks are given in section 4.

THE $\left(\frac{G'}{G}\right)$--EXPANSION METHOD

Here we briefly describe the main steps of the $\left(\frac{G'}{G}\right)$-expansion method. Consider a nonlinear partial differential equation (PDE) is of the form

$$P(u,u_t,u_x,\ u_{tt},\ u_{xt},u_{xx}........)=0, \tag{2}$$

where u=u(x,t) is an unknown function and P is polynomial in u= u(x,t) and its partial derivatives, in which higher order derivatives and nonlinear terms are involved. In order to solve eq.(2) by this method, one has to resort the following steps:

Step 1: To find the traveling wave solution of (2), introduce the wave variable $\xi = (x - ct)$, so that $u(x,t) = u(\xi)$.

Based on this,

$$\frac{\partial}{\partial t}=-c\frac{\partial}{\partial\xi},\quad \frac{\partial^2}{\partial t^2}=c^2\frac{\partial^2}{\partial\xi^2},\quad \frac{\partial}{\partial x}=\frac{\partial}{\partial\xi},\quad \frac{\partial^2}{\partial x^2}=\frac{\partial^2}{\partial\xi^2}, \tag{3}$$

and so on for other derivatives. With the help of (3), the PDE (2) changes to an ordinary differential equation (ODE) as

$$P(u,u_\xi,u_{\xi\xi},\ u_{\xi\xi\xi},........)=0, \tag{4}$$

where $u_\xi, u_{\xi\xi}$ etc. denote derivative of u with respect to ξ. Now integrate the ODE (4) as many times as possible and set the constants of integration to be zero.

Step 2: The solution of (4) can be expressed by a polynomial in $\left(\frac{G'}{G}\right)$ as

$$u(\xi)=\alpha_m\left(\frac{G'}{G}\right)^m+\alpha_{m-1}\left(\frac{G'}{G}\right)^{m-1}+........., \tag{5}$$

where $G = G(\xi)$ satisfies the second order linear ODE of the form

$$G''+\lambda G'+\mu G=0 \tag{6}$$

where $\alpha_m,\alpha_{m-1},.......,\alpha_0,\lambda$ and μ are constants to be determined later and $\alpha_m \neq 0$. The positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in ODE (4), after using (5).

Step 3: Substituting (5) into (4) and using (6), collecting all terms with the same order of $(\frac{G'}{G})$ together, and then equating each coefficient of the resulting polynomial to zero yields a set of algebraic equations for $\alpha_m, \alpha_{m-1}, \ldots, \alpha_0, \lambda, c$ and μ.

Step 4: Substituting α_i (i=0,1,2,....,m), c, λ, μ obtained in step 3 and the general solutions of eq.(6) into (5), we can obtain traveling wave solutions of the nonlinear PDE (2). The general solutions of (6) are given as

$$
\left(\frac{G'}{G}\right) = \begin{cases} \dfrac{\sqrt{\lambda^2-4\mu}}{2}\left(\dfrac{A_1\sinh(\frac{1}{2}\sqrt{\lambda^2-4\mu})\xi + A_2\cosh(\frac{1}{2}\sqrt{\lambda^2-4\mu})\xi}{A_1\cosh(\frac{1}{2}\sqrt{\lambda^2-4\mu})\xi + A_2\sinh(\frac{1}{2}\sqrt{\lambda^2-4\mu})\xi}\right) - \dfrac{\lambda}{2}, & \lambda^2-4\mu \succ 0, \end{cases}
$$

$$
= \begin{cases} \dfrac{\sqrt{4\mu-\lambda^2}}{2}\left(\dfrac{-A_1\sin(\frac{1}{2}\sqrt{4\mu-\lambda^2})\xi + A_2\cos(\frac{1}{2}\sqrt{4\mu-\lambda^2})\xi}{A_1\cos(\frac{1}{2}\sqrt{4\mu-\lambda^2})\xi + A_2\sin(\frac{1}{2}\sqrt{4\mu-\lambda^2})\xi}\right) - \dfrac{\lambda}{2}, & \lambda^2-4\mu \prec 0, \end{cases} \quad (7)
$$

$$
= \begin{cases} \dfrac{A_2}{A_1+A_2\xi} - \dfrac{\lambda}{2}, & \lambda^2-4\mu = 0, \end{cases}
$$

The above results can further be written in some more simplified forms [11] depending upon the conditions on the ratio of A_1 and A_2 as

$$
\left(\frac{G'}{G}\right) = \begin{cases} \dfrac{\sqrt{\lambda^2-4\mu}}{2}\tanh(\dfrac{\sqrt{\lambda^2-4\mu}}{2}\xi+\xi_0) - \dfrac{\lambda}{2}, & \lambda^2-4\mu \succ 0, \tanh(\xi_0) = \dfrac{A_1}{A_2}, \left|\dfrac{A_1}{A_2}\right| \succ 1, \end{cases}
$$

$$
= \begin{cases} \dfrac{\sqrt{\lambda^2-4\mu}}{2}\coth(\dfrac{\sqrt{\lambda^2-4\mu}}{2}\xi+\xi_0) - \dfrac{\lambda}{2}, & \lambda^2-4\mu \succ 0, \coth(\xi_0) = \dfrac{A_1}{A_2}, \left|\dfrac{A_1}{A_2}\right| \prec 1, \end{cases}
$$

$$
= \begin{cases} \dfrac{\sqrt{4\mu-\lambda^2}}{2}\cot(\dfrac{\sqrt{4\mu-\lambda^2}}{2}\xi+\xi_0) - \dfrac{\lambda}{2}, & \lambda^2-4\mu \prec 0, \ \cot(\xi_0) = \dfrac{A_2}{A_1}, \end{cases}
$$

$$
= \begin{cases} \dfrac{A_1}{A_1+A_2\xi} - \dfrac{\lambda}{2}, & \lambda^2-4\mu = 0. \end{cases} \quad (8)
$$

Therefore, these results are the simplified form of result obtained by $\left(\frac{G'}{G}\right)$-method. Hence we call this method is the simplified $\left(\frac{G'}{G}\right)$-method.

Now substituting $\alpha_m, \alpha_{m-1}, \ldots, \alpha_0$, c and the general solutions of (6) which are from (7) and (8) into (5) ,we obtain more traveling wave solutions of nonlinear differential equation (2).

After the brief description of method, we now solve the Bogoyavlenskii equation using the above methods.

EXACT SOLUTIONS TO THE BOGOYAVLENSKII EQUATION

In the theory of nonlinear waves, one of the most important aspects is the study of traveling wave solutions which are solutions of constant form moving with a fixed velocity. The traveling wave solution for eq. (1) is of the form

$$u(x,y,t)=u(\xi),\ v(x,y,t)=v(\xi),\quad \xi=x+y-ct. \tag{9}$$

Substituting eq.(9) into eq.(1) and integrating once the second equation of eq. (1) and equating the constant of integration is equal to zero, then we get

$$\begin{aligned}&-4cu'+u'''-4u^2u'-4u'v=0,\\&\frac{u^2}{2}=v.\end{aligned} \tag{10}$$

The substitution of the second equation of eq. (10) into the first equation, after integrating once the resultant, yields

$$u''-2u^3-4cu=0 \tag{11}$$

Now, balancing u'' with u^3 in (11), we get m=1. Then we suppose that

$$u(\xi)=\alpha_1(\frac{G'}{G})+\alpha_0,\quad \alpha_1\neq 0 \tag{12}$$

where $G = G(\xi)$ satisfies the second order linear ODE (6).

By substituting (12) into (10) and collecting all terms with the same power of $\left(\frac{G'}{G}\right)$- together, the left- hand sides of (10) are converted into the polynomials in $\left(\frac{G'}{G}\right)$. Equating each coefficient of the polynomials to zero, yields a set of simultaneous algebraic equations as

$$2\alpha_1-b\alpha_1^3=0, \tag{13}$$

$$(3\lambda-c)\alpha_1-3b\alpha_1^2\alpha_0=0, \tag{14}$$

$$(2\mu-c\lambda+\lambda^2+a)\alpha_1-3b\alpha_1\alpha_0^2=0, \tag{15}$$

$$(\lambda-\mathrm{c})\alpha_1+\mathrm{a}\alpha_0-\mathrm{b}\alpha_0^3=0. \tag{16}$$

which on solving gives

$$\alpha_1=\pm 1,\quad \alpha_0=\pm\frac{\lambda}{2},\quad c=-\frac{1}{8}(\lambda^2-4\mu) \tag{17}$$

Now, substituting (17) into (12) and the general solution of second order Linear ODE (6) into (12), we have three types of traveling wave solutions of (1) as follows:

Case 1: When $\lambda^2-4\mu>0$, the hyperbolic traveling wave solutions are given as

$$u(\xi)=\pm\frac{\sqrt{\lambda^2-4\mu}}{2}\left(\frac{A_1\sinh(\frac{1}{2}\sqrt{\lambda^2-4\mu})\xi+A_2\cosh(\frac{1}{2}\sqrt{\lambda^2-4\mu})\xi}{A_1\cosh(\frac{1}{2}\sqrt{\lambda^2-4\mu})\xi+A_2\sinh(\frac{1}{2}\sqrt{\lambda^2-4\mu})\xi}\right), \tag{18}$$

$$v(\xi)=\frac{(\lambda^2-4\mu)}{8}\left(\frac{A_1\sinh(\frac{1}{2}\sqrt{\lambda^2-4\mu})\xi+A_2\cosh(\frac{1}{2}\sqrt{\lambda^2-4\mu})\xi}{A_1\cosh(\frac{1}{2}\sqrt{\lambda^2-4\mu})\xi+A_2\sinh(\frac{1}{2}\sqrt{\lambda^2-4\mu})\xi}\right)^2, \tag{18a}$$

where $\xi = x + y + \frac{1}{8}(\lambda^2 - 4\mu)t$.

In particular, if $A_1 \neq 0,\ A_2 = 0,\ \lambda \succ 0,\ \mu = 0$, then $u(\xi)$ becomes

$$u(\xi) = \pm\frac{\lambda}{2}\tanh(\frac{\lambda}{2}\xi). \tag{19}$$

$$v(\xi) = \frac{\lambda^2}{8}\tanh^2(\frac{\lambda}{2}\xi). \tag{19a}$$

But, if $A_2 \neq 0,\ A_1 = 0,\ \lambda \succ 0,\ \mu = 0$, then $u(\xi)$ becomes

$$u(\xi) = \pm\frac{\lambda}{2}\coth h(\frac{\lambda}{2}\xi). \tag{20}$$

$$v(\xi) = \frac{\lambda^2}{8}\coth h^2(\frac{\lambda}{2}\xi). \tag{20a}$$

Again using (8), we derive the general solution $u(\xi)$ in simplified form as

$$u(\xi) = \pm\frac{\sqrt{\lambda^2 - 4\mu}}{2}\tanh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\xi + \xi_0\right), \tag{21}$$

$$v(\xi) = \frac{(\lambda^2 - 4\mu)}{8}\tanh^2\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\xi + \xi_0\right), \tag{21a}$$

when $\left|\frac{A_2}{A_1}\right| \succ 1,$ and $\xi_0 = \tanh^{-1}\frac{A_2}{A_1}$.

But when $\left|\frac{A_2}{A_1}\right| \prec 1,$ then $u(\xi)$ become

$$u(\xi) = \pm\frac{\sqrt{\lambda^2 - 4\mu}}{2}\coth\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\xi + \xi_0\right), \tag{22}$$

$$v(\xi) = \pm\frac{(\lambda^2 - 4\mu)}{8}\coth^2\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\xi + \xi_0\right), \tag{22a}$$

Where $\xi_0 = \tanh^{-1}\frac{A_2}{A_1}$.

Case 2: When $\lambda^2 - 4\mu \prec 0$, we get trigonometric solutions

$$u(\xi) = \pm\frac{\sqrt{4\mu - \lambda^2}}{2}\left(\frac{-A_1\sin(\frac{1}{2}\sqrt{4\mu - \lambda^2})\xi + A_2\cos(\frac{1}{2}\sqrt{4\mu - \lambda^2})\xi}{A_1\cos(\frac{1}{2}\sqrt{4\mu - \lambda^2})\xi + A_2\sin(\frac{1}{2}\sqrt{4\mu - \lambda^2})\xi}\right), \tag{23}$$

$$v(\xi) = \frac{4\mu - \lambda^2}{8}\left(\frac{-A_1\sin(\frac{1}{2}\sqrt{4\mu - \lambda^2})\xi + A_2\cos(\frac{1}{2}\sqrt{4\mu - \lambda^2})\xi}{A_1\cos(\frac{1}{2}\sqrt{4\mu - \lambda^2})\xi + A_2\sin(\frac{1}{2}\sqrt{4\mu - \lambda^2})\xi}\right)^2, \tag{23a}$$

The above result can be written in simplified forms as

$$u(\xi) = \pm \frac{\sqrt{4\mu - \lambda^2}}{2} \cot\left(\frac{\sqrt{4\mu - \lambda^2}}{2} \xi + \xi_0 \right), \tag{24}$$

$$v(\xi) = \frac{4\mu - \lambda^2}{8} \cot\left(\frac{\sqrt{4\mu - \lambda^2}}{2} \xi + \xi_0 \right)^2, \tag{24a}$$

where $\xi_0 = \cot^{-1} \frac{A_2}{A_1}$.

Case 3: When $\lambda^2 - 4\mu = 0$, the rational solutions are given as

$$u(\xi) = \pm \frac{A_2}{A_1 + A_2\xi} \tag{25}$$

$$v(\xi) = \frac{1}{2}\left(\frac{A_2}{A_1 + A_2\xi} \right)^2 \tag{25}$$

These are traveling wave solutions of the Bogoyavlenskii equation (1) under different assumption. Some of these solutions are reduced to solutions obtained by other methods under some considerations and others are new solutions.

CONCLUSION

In this work we obtained exact solutions of the Bogoyavlenskii equation are studied by the $\left(\frac{G'}{G}\right)$-- expansion method. The general traveling wave solutions can give soliton or periodic solutions under different parametric restrictions. We have also derived the general results of the above mentioned systems by applying the simplified $\left(\frac{G'}{G}\right)$--expansion method. It is interesting to note that from the general results, one can easily recover numerous solutions which are obtained by others methods.. This direct and concise method can further be used to explore more applications.

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