IAAST
ONLINE ISSN 2277- 1565
PRINT ISSN 0976 - 4828
International Archive of Applied Sciences and Technology
IAAST; Vol 4 [3] September 2013: 58-61

[ISO9001: 2008 Certified Organization]
www.soeagra.com/iaast/iaast.htm
CODEN: IAASCA
ORIGINAL ARTICLE

Gaussian Integer Solutions of Homogeneous Quadratic Equation with Four Unknowns $x^2 + y^2 = 3z^2 + w^2$

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ABSTRACT
The homogeneous quadratic equation with four unknowns represented by the diophantine equation $x^2 + y^2 = 3z^2 + w^2$ has been analyzed for Gaussian integer solutions.A few interesting relations between the solutions are exhibited.
KEYWORDS: Gaussian integers, homogeneous quadratic equation
M.Sc 2000 mathematics subject classification: 11D25

Received 11.08.2013 Accepted 04.09.2013

NOTATIONS

$t_{m,n}$: Polygonal number of rank n with size m

INTRODUCTION

The theory of diophantine equations offer a rich variety of fascinating problems since antiquity [1,2].In particular,Gaussian integer solutions have been analysed for special ternary quadratic diophantine equations [3-6]. This communication concerns with the yet another interesting homogeneous quadratic equation given by $x^2 + y^2 = 3z^2 + w^2$ for determining its infinitely many non-zero gaussian integral points.Also,a few interesting realations among the solutions are presented.

METHOD OF ANALYSIS

The quadratic equation with four unknowns to be solved is

$$x^2 + y^2 = 3z^2 + w^2 \quad (1)$$

Introduction of the linear transformations

$$x = u + v, y = u - v, z = v \quad (2)$$

in (1) leads to

$$v^2 + w^2 = 2u^2 \quad (3)$$

Again,the substitution of the linear transformations

$$v = 2A + i(r + s), w = 2A + i(r - s), u = 2r + iA \quad (4)$$

in (3) leads to $$5A^2 = 5r^2 + s^2 \quad (5)$$

Taking $A = \alpha + \beta , r = \alpha - \beta$ (6)

in (5),it simplifies to $20\alpha\beta = s^2$ (7)

which is satisfied by

$$\left.\begin{array}{l}\alpha = 5^{2k+1}\beta \\ \beta = 10(5^k)\beta, \beta > 0\end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (8)$$

Using (8),(6),(4) in (2),the corresponding gaussian integral solutions of (1) are represented by

$$x = 4\beta(5^{2k+1}) + 2i\beta(5^{k+1})(5^k + 1)$$
$$y = -4\beta + 2i\beta(1 - 5^{k+1})$$
$$z = 2\beta(5^{2k+1} + 1) + i\beta\left[5^{k+1}(5^k + 2) - 1\right]$$
$$w = 2\beta(5^{2k+1} + 1) + i\beta\left[5^{k+1}(5^k - 2) - 1\right]$$

Properties:

1. $5^k y + x \equiv 0 (\bmod 4)$
2. $x - z - w \equiv 0 (\bmod 2\beta)$
3. $x = y + 2z$
4. $x - y - 2w \equiv 0 (\bmod 40)$
5. $6\left[\dfrac{20(x + 5^{2k+1})}{w - z} + 6\right]$ is a nasty number [7]
6. $\dfrac{z^2 + w^2 - 2zw}{5^{2k}} + 800 t_{3,\beta} \equiv 0 (\bmod \beta)$

However,We have four more patterns of gaussian integer solutions for (1) which are illustrated below.

Pattern:1

Assume $A = a^2 + 5b^2$ (9)

Write 5 as $5 = (i\sqrt{5})(-i\sqrt{5})$ (10)

Substituting (10),(9) in (5) and employing the method of factorization,define

$$(s + i\sqrt{5}r) = (i\sqrt{5})(a + i\sqrt{5}b)^2 \quad (11)$$

Equating real and imaginary parts, we get

$$\left.\begin{array}{l}s = -10ab \\ r = a^2 - 5b^2\end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (12)$$

In view of (12),(10),(9),(4) and (2),the corresponding non-zero distinct gaussian integer solutions of (1) are obtained as

$$x = 4a^2 + i(2a^2 - 10ab)$$
$$y = -20b^2 + 10i(b^2 + ab)$$
$$z = 2a^2 + 10b^2 + i(a^2 - 5b^2 - 10ab)$$
$$w = 2a^2 + 10b^2 + i(a^2 - 5b^2 + 10ab)$$

Pattern:2

Rewrite (5) as $5A^2 - s^2 = 5r^2$ (13)

Let $r = 5a^2 - b^2, a, b > 0$ (14)

and write 5 as $5 = \frac{(3\sqrt{5}+5)(3\sqrt{5}-5)}{4}$ (15)

Following the similar calculations,we get

$$\left.\begin{aligned} A &= \frac{1}{2}(15a^2 + 3b^2 + 10ab) \\ s &= \frac{1}{2}(25a^2 + 5b^2 + 30ab) \end{aligned}\right\} \qquad (16)$$

Choosing $a = 2a, b = 2b$ in (16) and using (4) in (2), the corresponding non-zero distinct gaussian integeral solutions of (1) are as follows:

$$x = 100a^2 + 4b^2 + 40ab + 4i(25a^2 + 3b^2 + 20ab)$$
$$y = -20(a^2 + b^2 + 2ab) - 40i(a^2 + ab)$$
$$z = 60a^2 + 12b^2 + 40ab + 2i(35a^2 + 3b^2 + 30ab)$$
$$w = 60a^2 + 12b^2 + 40ab - 2i(15a^2 + 7b^2 + 30ab)$$

Pattern:3

(5) can be written in the form of ratio as

$$\frac{5(A+r)}{s} = \frac{s}{A-r} = \frac{P}{Q}, Q > 0 \qquad (17)$$

which is equivalent to the system of double equations

$$\left.\begin{aligned} 5QA + 5Qr - Ps &= 0 \\ -PA + \Pr + Qs &= 0 \end{aligned}\right\} \qquad (18)$$

Applying the method of cross mutiplication, we get

$$\left.\begin{aligned} A &= 5Q^2 + P^2 \\ r &= P^2 - 5Q \\ s &= 10PQ \end{aligned}\right\} \qquad (19)$$

In view of (19),(4) and (2),the corresponding non-zero gaussian integral solution\s of (1) are giuven by

$$x = 4P^2 + 2i(P^2 + 5PQ)$$
$$y = -20Q^2 + 10i(Q^2 - PQ)$$
$$z = 10Q^2 + 2P^2 + i(P^2 - 5Q^2 + 10PQ)$$
$$w = 10Q^2 + 2P^2 + i(P^2 - 5Q^2 - 10PQ)$$

Pattern:4

Similarly, Note that (5) is also written in the form of ratio as

$$\frac{(A+r)}{s} = \frac{s}{5(A-r)} = \frac{P}{Q}, Q > 0 \tag{20}$$

Following the procedure,as in Pattern (3),the corresponding another gaussian integral solutions of (1) are given by

$$x = 20P^2 + 10i(P^2 + pQ)$$
$$y = -4Q^2 + 2i(Q^2 - 5PQ)$$
$$z = 2Q^2 + 10P^2 + i(5P^2 - Q^2 + 10PQ)$$
$$w = 2Q^2 + 10P^2 + i(5P^2 - Q^2 - 10PQ)$$

CONCLUSION

In this paper, we have presented in total five different patterns of non-zero distinct gaussian integral solutions to (1).One may search for other pattern of solutions and their corresponding properties.

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Citation of Article: M.A.Gopalan, G.Sumathi , S.Vidhyalakshmi. Gaussian Integer Solutions of Homogeneous Quadratic Equation with Four Unknowns $x^2 + y^2 = 3z^2 + w^2$. Int. Arch. App. Sci. Technol., Vol 4 [3] September 2013: 58-61