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ORIGINAL ARTICLE

On The Homogeneous Bi-quadratic Equation with Five Unknowns $x^4 - y^4 = 5(z^2 - w^2)R^2$

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ABSTRACT

The Bi-quadratic Equation with 5 unknown given by $x^4 - y^4 = 5(z^2 - w^2)R^2$ is analyzed for its patterns of non – zero distinct integral solutions. A few interesting relations between the solutions and special polygonal numbers are exhibited.

Keywords: *Quadratic equation, Integral solutions, Special polygonal numbers, Pyramidal numbers.*

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INTRODUCTION

Bi-quadratic Diophantine Equations, homogeneous and non- homogeneous, have aroused the interest of numerous Mathematicians since ambiguity as can be seen from [1-7]. In the context one may refer [8-20] for varieties of problems on the Diophantine equations with two, three and four variables. This communication concerns with the problems of determining non-zero integral solutions of yet another quadratic equation in 5 unknowns represented by $x^4 - y^4 = 5(z^2 - w^2)R^2$. A few interesting relations between the solutions and special polygonal numbers are presented.

NOTATIONS USED

- $t_{m,n}$ - Polygonal number of rank n with size m .
- P_n^m - Pyramidal number of rank n with size m .
- $ct_{m,n}$ - Centered polygonal number of rank n with size m .
- gn_a - Gnomonic number of rank a
- so_n - Stella octangular number of rank n
- s_n - Star number of rank n
- pr_n - Pronic number of rank n
- pt_n - Pentatope number of rank n
- $CP_{m,n}$ - Centered pyramidal number of rank n with size m

METHOD OF ANALYSIS

The Diophantine equation representing the bi-quadratic equation with five unknowns under consideration is

$$x^4 - y^4 = 5(z^2 - w^2)R^2 \quad (1)$$

The substitution of the linear transformations

$$x = u + v,\ y = u - v,\ z = 2u + v,\ w = 2u - v \quad (2)$$

in (1) leads to $u^2 + v^2 = 5R^2$ (3)

Different patterns of solutions of (1) are presented below

Pattern -1

Assume $R = a^2 + b^2$ where a and b are non-zero distinct integers. (4)

Write 5 as $5 = (2+i)(2-i)$ (5)

Using (4) & (5) in (3) and employing the method of factorization, define

$$u + iv = (2+i)(a+ib)^2$$

Equating the real and imaginary parts, we get

$$u = u(a,b) = 2a^2 - 2b^2 - 2ab$$
$$v = v(a,b) = a^2 - b^2 + 4ab$$

Hence in view of (2) the corresponding solutions of (1) are

$$x = x(a,b) = 3a^2 - 3b^2 + 2ab$$
$$y = y(a,b) = a^2 - b^2 - 6ab$$
$$z = z(a,b) = 5a^2 - 5b^2$$
$$w = w(a,b) = 3a^2 - 3b^2 - 8ab$$
$$R = R(a,b) = a^2 + b^2$$

A few interesting properties observed are as follows:

1. $x(a,a(a+1)) - 3y(a,a(a+1)) = 40p_a^5$
2. $z(a,b) - 5y(a,b) \equiv 0 (\mathrm{mod}\,30)$
3. $x(a,(a+1)(a+2)) - w(a,(a+1)(a+2)) = 60p_a^3$
4. $z(a,b) + R(a,b) = Nastynumber - t_{4,2b}$
5. Each of the following represents a nasty number:

- $3\{y(a,2a^2-1) + R(a,2a^2-1) + 6SO_a\}$
- $75R(a,b) + 15z(a,b)$
- $z(a,a) - y(a,a)$

Pattern-2:

Instead of (4) write 5 as

$$5 = (1+2i)(1-2i) \quad (6)$$

Following a similar procedure as in pattern-1, the solutions for (3) are as follows

$$\left.\begin{aligned} u = u(a,b) = a^2 - b^2 - 4ab \\ v = v(a,b) = 2a^2 - 2b^2 + 2ab \end{aligned}\right\} \quad (7)$$

In view of (2) and (7) the solutions of (1) are obtained as

$$x = x(a,b) = 3a^2 - 3b^2 - 2ab$$
$$y = y(a,b) = -a^2 + b^2 - 6ab$$
$$z = z(a,b) = 4a^2 - 4b^2 - 6ab$$
$$w = w(a,b) = -10ab$$
$$R = R(a,b) = a^2 + b^2$$

Properties:

1. $x(a,b) + 3y(a,b) = 2w(a,b) \equiv 0 \pmod{20}$
2. $-z(a,2a^2+1) - 4y(a,2a^2+1) = 90(OH_a)$
3. $x(a,b) - y(a,b) - z(a,b) + w(a,b) = 0$
4. $x(a,a^2) + y(a,a^2) = 2(t_{4,a} - t_{4,a^2}) + CP_{6,2a}$
5. Each of the following represents a nasty number:

- $3\{-y(a,a) - R(a,a) - 2t_{4,a}\}$
- $-y(a,a) \, and \, -z(a,a)$

Pattern-3:

In addition to (4) and (6),

write 5 as $5 = \frac{1}{25}(11+2i)(11-2i)$

Following the procedure as in pattern-2, the solutions for (3) are as follows

$$u = u(a,b) = \frac{1}{5}(11a^2 - 11b^2 - 4ab)$$
$$v = v(a,b) = \frac{1}{5}(2a^2 - 2b^2 + 22ab)$$

Hence the corresponding solutions of (1) are

$$x = x(a,b) = \frac{1}{5}(13a^2 - 13b^2 + 18ab)$$
$$y = y(a,b) = \frac{1}{5}(9a^2 - 9b^2 - 26ab)$$
$$z = z(a,b) = \frac{1}{5}(24a^2 - 24b^2 + 14ab)$$
$$w = w(a,b) = \frac{1}{5}(20a^2 - 20b^2 - 30ab)$$

As our interest on finding integer solutions, we choose a and b suitably so that the values of x, y, z, w are integers.

Illustration I:

Let $a = 5A$ and $b = 5B$

Thus the corresponding solutions of (1) are

$$x = x(A,B) = 65A^2 - 65B^2 + 90AB$$
$$y = y(A,B) = 45A^2 - 45B^2 - 130AB$$
$$z = z(A,B) = 120A^2 - 120B^2 + 70AB$$
$$w = w(A,B) = 100A^2 - 100B^2 - 150AB$$
$$R = R(A,B) = 5A^2 + 5B^2$$

Illustration II:

Put $a = 5A + 2$ and $b = 5B + 1$

Hence the corresponding solutions of (1) are

$$x = x(A,B) = 65A^2 - 65B^2 + 70A + 10B + 90AB + 15$$
$$y = y(A,B) = 45A^2 - 45B^2 + 10A - 70B - 130AB - 5$$
$$z = z(A,B) = 120A^2 - 120B^2 + 110A - 20B + 70AB + 20$$
$$w = w(A,B) = 100A^2 - 100B^2 + 50A - 100B - 150AB$$
$$R = R(A,B) = 25A^2 + 25B^2 + 20A + 10B + 5$$

Properties:

1. $24x(a,4a-3) - 13z(a,4a-3) = 50t_{10,a}$
2. $9w(a,a^2+1) - 20y(a,a^2+1) = 100CP_{3,a}$
3. $30\{x(a,a+1) - y(a,a+1)\} = Nastynumber - 24t_{4,a+1} + 4(Ct_{22,a} - 1)$
4. $9R(b+1,b) - 5y(b+1,b) = 2t_{4,3b} + 26\Pr_b$
5. $6\{9x(a,a) - 13y(a,a)\}$ is a nasty number.

Pattern-4:

Rewrite (3) as $5R^2 - v^2 = u^2 * 1$ (8)

Write 1 as $1 = (\sqrt{5}+2)(\sqrt{5}-2)$ (9)

Let $u = 5a^2 - b^2$ (10)

Using (9) & (10) in (8) and employing the method of factorization, we write

$$\sqrt{5}R + v = (\sqrt{5}+2)(\sqrt{5}a + b)^2$$

Equating the rational and irrational parts, we have

$$R = R(a,b) = 5a^2 + b^2 + 4ab$$
$$v = v(a,b) = 10a^2 + 2b^2 + 10ab \qquad (11)$$

In view of (2) and (11), the solutions of (1) are obtained as

$$x = x(a,b) = 15a^2 + b^2 + 10ab$$
$$y = y(a,b) = -5a^2 - 3b^2 - 10ab$$
$$z = z(a,b) = 20a^2 + 10ab$$
$$w = w(a,b) = -4b^2 - 10ab$$
$$R = R(a,b) = 5a^2 + b^2 + 4ab$$

Properties:

1. $z(a,b) + w(a,b) = 2(x(a,b) + y(a,b)) \equiv 0(\bmod 4)$

2. $x(a,-1) - R(a,-1) = 4t_{7,a}$

3. $x(a,b) - y(a,b) - z(a,b) - t_{4,2b} \equiv 0(\bmod 10)$

4. Each of the following represents a nasty number:

- $3\{x(a,a) + y(a,a)\}$
- $x(a,a) + y(a,a) + z(a,a) + w(a,a)$

Pattern-5:

Instead of (9), write 1 as $1 = \frac{1}{4}(\sqrt{5}+1)(\sqrt{5}-1)$

Following the same procedure as in pattern-4, the solutions for (3) are as follows

$$R = R(a,b) = \frac{1}{2}(5a^2 + b^2 + 2ab)$$

$$v = v(a,b) = \frac{1}{2}(5a^2 + b^2 + 10ab) \qquad (12)$$

In view of (2) and (12), the solutions of (1) are

$$x = x(a,b) = \frac{1}{2}(15a^2 - b^2 + 10ab)$$

$$y = y(a,b) = \frac{1}{2}(5a^2 - 3b^2 - 10ab)$$

$$z = z(a,b) = \frac{1}{2}(25a^2 - 3b^2 + 10ab)$$

$$w = w(a,b) = \frac{1}{2}(15a^2 - 5b^2 - 10ab)$$

$$R = R(a,b) = \frac{1}{2}(5a^2 + b^2 + 2ab)$$

The values of x, y, z, w and R are integers when both a and b are of the same parity.

Case- I:

Consider $a = 2A$ and $b = 2B$

Thus the corresponding solutions of (1) are

$$x = x(A,B) = 30A^2 - 2B^2 + 20AB$$

$$y = y(A,B) = 10A^2 - 6B^2 - 20AB$$

$$z = z(A,B) = 50A^2 - 6B^2 + 20AB$$

$$w = w(A,B) = 30A^2 - 10B^2 - 20AB$$

$$R = R(A,B) = 10A^2 + 2B^2 + 4AB$$

Case- II:

Put $a = 2A+1$ and $b = 2B+1$

Hence the corresponding solutions of (1) are

$$x = x(A,B) = 30A^2 - 2B^2 + 40A + 8B + 20AB + 12$$
$$y = y(A,B) = 10A^2 - 6B^2 - 16B - 20AB - 4$$
$$z = z(A,B) = 50A^2 - 6B^2 + 60A + 4B + 20AB + 16$$
$$w = w(A,B) = 30A^2 - 10B^2 + 20A - 20B - 20AB$$
$$R = R(A,B) = 10A^2 + 2B^2 + 12A + 4B + 4AB + 4$$

Properties:

1. $x(a,b) + y(a,b) + z(a,b) + w(a,b) \equiv 0 \pmod 6$
2. $x(a,-1) + R(a,-1) = 4t_{7,a}$
3. $z(a(a+1),2a+1) - x(a(a+1),2a+1) - y(a(a+1),2a+1) - R(a(a+1),2a+1) = 24P_a^4$
4. $3R(b+1,b) - 3y(b+1,b) - 36t_{3,b}$ is a nasty number.

Pattern-6:

Introduction of the linear transformations

$$R = X + T \qquad v = X + 5T \qquad u = 2U \tag{13}$$

in (3) leads to $U^2 = X^2 - 5T^2$

which is satisfied by

$$X = r^2 + 5s^2$$
$$u = 2(r^2 - 5s^2)$$
$$T = 2rs$$

Substituting the above values of X, u and T in (13), the corresponding non-zero distinct integral solutions of (3) are given by

$$R = R(a,b) = r^2 + 5s^2 + 2rs$$
$$v = v(a,b) = r^2 + 5s^2 + 10rs$$

Thus the corresponding solutions of (1) are found to be

$$x = x(a,b) = 3r^2 - 5s^2 + 10rs$$
$$y = y(a,b) = r^2 - 15s^2 - 10rs$$
$$z = z(a,b) = 5r^2 - 15s^2 + 10rs$$
$$w = w(a,b) = 3r^2 - 25s^2 - 10rs$$
$$R = R(a,b) = r^2 + 5s^2 + 2rs$$

Properties:

1. $x(1,s) - w(1,s) = 2(Ct_{20,s} - 1)$
2. $x(r,s) + y(r,s) + z(r,s) + w(r,s) \equiv 0 \pmod{12}$
3. $x(r,r(r+1)) + R(r,r(r+1)) = t_{4,2r} + 6P_r^3$

4. $z(r,s) - R(r,s) \equiv 0 (\bmod 4)$

5. Each of the following represents a nasty number:

- $-y(r,r) - z(r,r)$
- $3\{x(r,s) + w(r,s) - y(r,s) - z(r,s)\}$

REMARKABLE OBSERVATIONS

I: $\left[\dfrac{2P_{x-1}^{5}}{t_{4,x-1}}\right]^4 - \left[\dfrac{36P_{y-2}^{3}}{S_{y-2}-1}\right]^4 \equiv 0 (\bmod 5)$

II: $\left\{5\left[\dfrac{4Pt_{z-3}}{P_{z-3}^{3}}\right]^2 - 5\left[\dfrac{6P_w^4}{t_{6,w+1}}\right]^2\right\}\left[\dfrac{t_{3,2w-1}}{gn_w}\right]^2 + \left[\dfrac{3P_y^3}{t_{3,y}}\right]^4$ is a bi-quadratic integer.

III: $30\left[\dfrac{4P_x^5}{Ct_{4,x}-1}\right]^4 - 30\left[\left[\dfrac{P_{y-1}^4}{t_{3,y-1}}\right] - \left[\dfrac{P_{y-1}^3}{t_{3,z}}\right]\right]^4 + 150\left[\dfrac{CP_{-w}}{t_{4,-w}}\right]^2\left[\dfrac{6P_{R-1}^4}{t_{3,2R-2}}\right]^2$ is a nasty number.

IV: If the non-zero integer quintuple $(x_0, y_0, z_0, w_0, R_0)$ is any solution of (1) then the quintuple $(x_n, y_n, z_n, w_n, R_n)$

where

$$
\begin{aligned}
x_n &= u_0 + \tilde{y}_{n-1}v_0 + 5\tilde{x}_{n-1}R_0 \\
y_n &= u_0 - \tilde{y}_{n-1}v_0 - 5\tilde{x}_{n-1}R_0 \\
z_n &= 2u_0 + \tilde{y}_{n-1}v_0 + 5\tilde{x}_{n-1}R_0 \\
w_n &= 2u_0 - \tilde{y}_{n-1}v_0 - 5\tilde{x}_{n-1}R_0 \\
R_n &= \tilde{y}_{n-1}R_0 + \tilde{x}_{n-1}v_0
\end{aligned}
$$

also satisfies (1).In the above , u_0, v_0, R_0 are the initial solutions of (3) and $(\tilde{x}_{n-1}, \tilde{y}_{n-1})$ is the solution of the pellian $y^2 = 5x^2 + 1$

Note:

In linear transformations (2), the variables z and w may also be represented by
$z = 2uv + 1$, $w = 2uv - 1$

Applying the procedure similar to that presented above in patterns 1 to 6, other choices of integer solutions of (1) are obtained.

CONCLUSION

To conclude, one may search for other patterns of solutions and their corresponding properties.

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