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# Free (unrestricted) Concepts: Definitions, Properties and the Mathematical Proofs of their Algebraic Structure and Lattice Order

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#### ABSTRACT

The content of this paper is absolutely mathematical, concerning the concepts, their structure and their order: semigroup, group, lattice, order relation of the lattice, ring and Boolean ring, Boolean algebra. Although, applications of the above mentioned "Mathematical Theory of Concepts", have been published by me several times till now, it is the first time that the basic mathematical structure (proofs included) is published. I call the concepts free or unrestricted, because: 1. "Concept is every assignment of a prototype to an icon, whatever may be the prototype and the icon" (Definition 1.), 2. the standardized concepts (that is, involving exactly defined objects) are just a subset of the "whole world" of concepts (assignments), 3. in my structure of concepts, not all the concepts are ordered, because I define symmetric-differences of concepts. **Key-words:** prototype, operations, semigroup, Set Theory, object, attribute, group, Boolean, ring, symmetric-difference, field, algebra, integral domain, zero divisors, laws of de Morgan.

### SEMIGROUP STRUCTURE

Let D be any text or dictionary or collection of terms from a natural or artificial language. It is our data from which the concepts come. Let's call C the set of all these concepts.

**Definition 1**. Concept is every assignment of a prototype to an icon, whatever may be the prototype and the icon. We call the prototype "object" and the icon "attribute". We symbolize a concept with a couple whose left part is the object and right part the attribute.

**<u>Conclusion</u>**: C is not empty, since: i. D is not empty, ii. <u>"every</u> assignment of <u>any</u> prototype to <u>any</u> icon gives rise to a concept".

Let's take, now, two such concepts, (1,1') and (W,W'), and define the operations  $\cup$  and  $\cap$ 

inside C.

**Definition 2.** (1,1')  $\cap_{\text{def.}}$  (W,W') = (1  $\cup$  W, I'  $\cap$  W'), where  $\cup$  and  $\cap$  are the usual operations

between sets union and intersection respectively.

**Definition 3.** (1,1')  $\cap_{\text{def.}}$  (W,W') = (1  $\cap$  W, I'  $\cup$  W').

The operations  $\cup$  and  $\cap$  are well-defined because:

a. Object and attribute, left and right part of the couple, are necessarily sets. The human mind thinks with sets either they have no members, one member or many.

b. The operations  $\cup$  and  $\cap$  are well-defined between sets.

c. The results  $(1 \cup W, I' \cap W')$  and  $(1 \cap W, 1' \cup W')$  are really concepts because: i.  $1 \cup W$  and  $1 \cap W$  are obviously sets of objects and ii.  $1' \cap W'$  and  $1' \cup W'$  are obviously sets of attributes.

# Properties of the operation $\cup\,.$

1. Commutative. Indeed,

 $(1,1') \stackrel{\cdot}{\cup} (\mathsf{W},\mathsf{W}') \stackrel{=}{\underset{\mathrm{def},2}{=}} (1 \cup \mathsf{W},\mathsf{I}' \cap \mathsf{W}') \stackrel{=}{\underset{*}{=}} (\mathsf{W} \cup 1,\mathsf{W}' \cap \mathsf{I}')$ 

= (W,W')  $\cup$  (1,1').

- \* because of the commutative property of the operations  $\cup$  and  $\cap$ .
- 2. Associative. Indeed,

$$\begin{array}{c} \mathbf{(1,1')} \stackrel{\bullet}{\cup} [(\mathsf{W},\mathsf{W}') \stackrel{\bullet}{\cup} (\mathsf{S},\mathsf{S}')] = \\ _{\mathrm{def.2}} = (1 \cup (\mathsf{W} \cup \mathsf{S}), \mathsf{I}' \cap (\mathsf{W}' \cap \mathsf{S}')) = \\ = (1 \cup (\mathsf{W} \cup \mathsf{S}), \mathsf{I}' \cap (\mathsf{W}' \cap \mathsf{S}')) = \\ _{**} ((1 \cup \mathsf{W}) \cup \mathsf{S}, (1' \cap \mathsf{W}') \cap \mathsf{S}') = \\ _{\mathrm{def.2}} \end{array}$$

$$(1 \cup W, 1' \cap W') \stackrel{\bullet}{\cup} (S, S') = [(1,1') \stackrel{\bullet}{\cup} (W,W')] \stackrel{\bullet}{\cup} (S,S').$$

- \*\* because of the associative property of the operations  $\cup$  and  $\cap$ .
- 3. We ask for the neutral element (X,Y) for the operation  $\cup$ , or equivalently:

$$(1,1') \cup (X,Y) = (X,Y) \cup (1,1') = (1,1'), \forall (1,1') \in C.$$

Because of the commutative property of the operation  $\stackrel{\bullet}{\cup}$  we have only  $(1,1') \stackrel{\bullet}{\cup} (X,Y) = (1,1') \stackrel{\bullet}{\underset{def 2}{=}}$ 

 $(1 \cup X, 1' \cap Y) = (1,1') = (1 \cup X = 1 \text{ and } 1' \cap Y = 1') \iff (X = \emptyset \text{ and } Y = \Omega')$ , since the

equality must hold  $\forall$  (1,1')  $\in$  C.

The symbol  $\emptyset$  stands for the empty set and the symbol  $\Omega'$  (omega capital with accent) for the set of all attributes existing in the data D.

So the neutral element of the operation  $\cup$  is the concept ( $\emptyset$ ,  $\Omega'$ ). In the "real" world it is accepted as a concept of definition 1 because, to the empty set, we can ascribe every attribute. Besides, in our structure we don't make distinction "real" and "imaginary" (or whatsoever) world. We don't have to say why, to the object  $\emptyset$ , we assign all the attributes of  $\Omega'$ ).

\*\*\* As the equality relation between concepts of definition 1 we accept the usual equality between ordered couples, that is:

 $(a, b) = (c, d) \Leftrightarrow_{def.} (a = c \text{ and } b = d).$ 

4. We ask for the symmetric (inverse) element (X,Y) of the concept (1,1'), or equivalently:

 $(1,1') \cup (X,Y) = (X,Y) \cup (1,1') = (\emptyset, \Omega').$ 

Because of the commutative property of the operation  $\overset{\bullet}{\cup}$ , we have only  $(1,1') \overset{\bullet}{\cup} (X,Y) = (\emptyset, \Omega') = (\emptyset, \Omega')$ 

 $(1 \cup X, 1' \cap Y) = (\emptyset, \Omega') \Leftrightarrow (1 = \emptyset \text{ and } X = \emptyset \text{ and } 1' = \Omega' \text{ and } Y = \Omega')$ . This means that the neutral element  $(\emptyset, \Omega')$  is the only concept with symmetric element (itself).

# Conclusion

The set C, enriched with the operation  $\,\cup\,,$  is a commutative (abelian) semigroup with neutral element.

Properties of the operation  $\,\cap\,$ 

Similarly we find that the set C, enriched with the operation  $\,\cap$  , is a commutative semigroup with

neutral element the concept  $(\Omega, \emptyset)$ , where  $\Omega$  is the set of all objects coming from the data D. As it was said about the neutral element  $(\emptyset, \Omega')$ , here again we say that the couple  $(\Omega, \emptyset)$  is accepted as a concept of definition 1 since: i. the set  $\Omega'$  of attributes is a right-part element, ii. the set  $\Omega$  of objects is a left-part element, iii. the empty set  $\emptyset$  can be easily seen as the empty set of objects or the empty set of attributes.

# LATTICE STRUCTURE

Between the concepts of C, the following properties hold:

- 1. Commutative for both operations  $\cup$  and  $\cap$  (already proved).
- 2. Associative for both operations (also proved).

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3. Absorbent (property of absorption) for both operations.

We give here the proof for the operation  $\dot{\cup}$  (and similar is the proof for  $\cap$ ):

$$[(0_1, A_1) \cap (0_2, A_2)] \stackrel{\bullet}{\cup} (0_1 \cap 0_2, A_1 \cup A_2,) \\ \stackrel{\bullet}{\cup} (0_1, A_1) \stackrel{=}{\underset{def, 2}{\longrightarrow}} ((0_1 \cap 0_2) \cup 0_1, (A_1 \cup A_2) \cap A_1,) \stackrel{=}{\underset{*}{\longrightarrow}} (0_1, A_1).$$

\* because of the absorption property between the subsets of a given set X (as it is known from the elementary Set Theory).

From the Classic Theory of Lattices we know that the holding of the properties 1., 2. and 3. is equivalent to the structure of lattice. Therefore, <u>C is a lattice</u>.

Moreover, <u>C is a distributive lattice</u>. We give here the proof that  $\cap$  is distributive to  $\cup$  (and

similar is the proof for  $\stackrel{\bullet}{\cup}$  to  $\stackrel{\bullet}{\cap}$ ):

$$(O_{1}, A_{1}) \cap [(O_{2}, A_{2}) \cup (O_{3}, A_{3})] = (O_{1}, A_{1}) \cap (O_{2} \cup O_{3}, A_{2} \cap A_{3}) =$$

$$= (O_{1} \cap (O_{2} \cup O_{3}), A_{1} \cup (A_{2} \cap A_{3})) = ((O_{1} \cap O_{2}) \cup (O_{1} \cap O_{3}), (O_{1} \cup A_{2}) \cap (A_{1} \cup A_{3})) = (O_{1} \cap O_{2}, A_{1} \cup A_{2}) \cup (O_{1} \cap O_{3}, A_{1} \cup A_{3}) =$$

$$= [(O_{1}, A_{1}) \cap (O_{2}, A_{2})] \cup [(O_{1}, A_{1}) \cap (O_{3}, A_{3})].$$

\*\* because the distributive properties of the usual operations union ( $\cup$ ) and intersection ( $\cap$ ) hold between subsets of a given set X.

# Order relation in the lattice C.

From the Classic Theory of Lattices we know that we can define the order relation  $\leq$  (here let's use

the symbol  $\underline{\frown}$ ) of the lattice out of the already existing operations v (here  $\bigcirc$ ) and  $\Lambda$  (here  $\frown$ ).

Definition 4.  $(O_1, A_1) \xrightarrow{\frown} (O_2, A_2) \Leftrightarrow (O_1, A_1) \cap (O_2, A_2) = (O_1, A_1).$ 

This is the classic definition of the order in a lattice but now, using definition 3, we take:  $(O_1, A_1) \cap (O_2, A_2) = (O_1, A_1) \underset{\text{def. 3}}{\Leftrightarrow} (O_1 \cap O_2, A_1 \cup A_2) =$ 

$$= (O_1, A_1) \Leftrightarrow (O_1 \cap O_2 = O_1 \text{ and } A_1 \cup A_2 = A_1) \Leftrightarrow_{***} (O_1 \subseteq O_2) \text{ and }$$

 $A_1 \supseteq A_2$ ).

Combining this last result and the definition 4, we take an equivalent definition of the order relation  $\bigcirc$  (let's call it definition 4a):

Definition 4a. (01, A1)  $\subset$  (0<sub>2</sub>, A<sub>2</sub>)  $\Leftrightarrow$  (0<sub>1</sub>  $\subset$  0<sub>2</sub> and A1  $\supset$  A<sub>2</sub>).

\*\* because of the usual order relation  $\_$  (and its dual  $\_$ ) between subsets of a given set X. These

subsets form the lattice (P(X),  $\cup$ ,  $\cap$ ), with behaviour exactly the same as of the lattice (C,  $\cup$ ,  $\cap$ ).

Remark. Though in the Classic Theory of Lattices it is proved that definition 4 gives really an order relation, we shall prove here the three properties of ordering for the relation  $\subset$ .

1. Reflexive. Indeed,  $(0, A) \subseteq (0, A) \Leftrightarrow_{def.4a} (0 \subseteq 0 \text{ and } A \supset A)$  which do hold (reflexivity of  $\subseteq$  and  $\supseteq$ ). 2. Antisymmetric. Indeed,

 $\begin{array}{l} (O_{1},A_{1}) \underline{\frown} (O_{2},A_{2}) \underset{def.4a}{\Leftrightarrow} (O_{1} \underline{\frown} O_{2} \text{ and } A_{1} \underline{\supset} A_{2}) \mbox{(2a)} \\ (O_{2},A_{2}) \underline{\frown} (O_{1},A_{1}) \underset{def.4a}{\Leftrightarrow} (O_{2} \underline{\frown} O_{1} \text{ and } A_{2} \underline{\supset} A_{1}) \mbox{(2b)}. \end{array}$ 

From the equivalencies (2a) and (2b) we take  $[(O_1 \subseteq O_2 \text{ and } O_2 \subseteq O_1) \text{ and } (A_1 \subseteq A_2 \supseteq A_1)] \underset{****}{\Leftrightarrow} (O_1 \otimes O_2 \otimes O_2)$ 

 $= O_2 \text{ and } A_1 = A_2) \iff (O_1, A_1) = (O_2, A_2).$ 

\*\*\*\* because of the antisymmetric property of the usual order  $\_$ .

3. Transitive. Indeed,

 $\begin{array}{l} (O_1, A_1) \underline{\subset} (O_2, A_2) \underset{def.4a}{\Leftrightarrow} (O_1 \underline{\subset} O_2 \text{ and } A_1 \underline{\supset} A_2) \text{ (3a)} \\ (O_2, A_2) \underline{\subset} (O_3, A_3) \underset{def.4a}{\Leftrightarrow} (O_2 \underline{\subset} O_3 \text{ and } A_2 \underline{\supset} A_3) \text{ (3b)} \end{array}$ 

From the equivalencies (3a) and (3b) and because of the transitive property of the usual order relation  $\subset$ , we take:

$$(O_1 \subseteq O_3 \text{ and } A_3 \supseteq A_1) \underset{\text{def} 4a}{\Leftrightarrow} (O_1, A_1) \subseteq (O_3, A_3).$$

The lattice (C,  $\bigcup$ ,  $\bigcirc$ ) is complete.

A lattice C is complete if, for every subset A of L ( $\emptyset \neq A \subseteq L$ ), the supremum and infimum exist in L. Let's take such a set A  $\subseteq$  C. Indeed, there is an infimum and a supremum for A, which may belong or not to A but surely they belong to C. Since A  $\subseteq$  C, it consists of concepts, that is of couples. The couples of A belong also to C, with the set-theoretic meaning of  $\in$ , and so we use the order  $\subseteq$ between the subsets of C (that is, in the set P(C)) and not the order  $\subseteq$  between the couples of C (that, is in the lattice C).

The couples of A have, as left-part elements, sets of objects and, as right-part elements, sets of attributes. The union  $(\cup)$  of all left-part elements is a set  $\cup_L$  of objects and it may be a left-part element of a couple of A or may not. But surely it belongs  $(\in)$  to  $P(\Omega)$ , that is, it is a subset of  $\Omega$ . Hence it is a left-part element of a couple in C. The intersection  $(\cap)$  of all right-part elements is a set  $\cap_R$  of attributes and it may be a right-part element of a couple of A or may not. But surely it belongs  $(\in)$  to  $P(\Omega)$ , that is, it is a subset of  $\Omega$ . ( $\in$ ) to  $P(\Omega')$ , that is, it is a subset of  $\Omega'$ . Hence, it is a right-part element of a couple in C.

Moreover,  $\cup_L \supseteq$  every left-part element since it has come from the union ( $\cup$ ) of these left-part elements and  $\cap_R \subseteq$  every right-part element since it has come from the intersection ( $\cap$ ) of these right-part elements. Consequently,  $(\cup_L, \cap_R) :\supseteq$  every couple of A (definition 4a between concept couples) and this means that  $(\cup_L, \cap_R)$  is the supremum of the set A.

Dually, the infimum is the couple  $(\cap_L, \cup_R)$  (the meaning of the symbols  $\cup_L$  and  $\cap_R$  is obvious).

# **RING AND BOOLEAN RING STRUCTURE**

Definition 5. The complement of the concept (O, a)  $\in$  C is the concept (O<sup>c</sup>, A<sup>c</sup>)  $\in$  C where O<sup>c</sup> and A<sup>c</sup> are the usual set-theoretic complements of O and A, referring to  $\Omega$  and  $\Omega'$ , respectively. In symbols: (O, A)<sup>c</sup> = (O<sup>c</sup>, A<sup>c</sup>).

The complement of a concept is well-defined because:

a.  $O^{c} \subset \Omega$  and  $A^{c} \subset \Omega'$ . Therefore  $(O^{c}, A^{c}) \in C$ .

b. There is only one complement  $O^c$  of O and only one  $A^c$  of A. Therefore there is only one complement  $(O, A)^{c} = (O^c, A^c)$  of the concept (O, A).

Properties of the complement C..

1. (0, A)  $\stackrel{\bullet}{\cup}$  (0, A)<sup>c</sup> = (0, A)  $\stackrel{\bullet}{\cup}$  0<sup>c</sup>, A<sup>c</sup>) = (0  $\cup$  0<sup>c</sup>, A  $\cap$  A<sup>c</sup>) = ( $\Omega$ ,  $\emptyset$ ), which is the neutral element of the operation  $\cap$ . This is exactly the same in the usual Set Theory where X  $\cup$  X<sup>c</sup> =  $\Omega$ ,  $\Omega$  being the neutral element of the operation  $\cap$ .

2. (0, A)  $\cap_{\bullet}$  (0, A)<sup>c</sup> = (0, A)  $\cap_{\bullet}$  (0<sup>c</sup>, A<sup>c</sup>) = (0  $\cap$  0<sup>c</sup>, A  $\cup$  A<sup>c</sup>) = ( $\emptyset$ ,  $\Omega$ '), which is the neutral

element of the operation  $\cup$ . The analogous in the usual Set Theory is  $X \cap X^c = \emptyset$ , where  $\emptyset$  is the neutral element of the operation  $\cup$ .

3.  $((O, A)^{c})^{c} = (O^{c}, A^{c})^{c} = ((O^{c})^{c}, (A^{c})^{c}) = (O, A)$ , which is exactly the same with the equality (\*)  $(X^{c})^{c} = X$  of the usual Set Theory.

4.  $(\emptyset, \Omega')^{c} = (\emptyset^{c}, \Omega'^{c}) = (\Omega, \emptyset)$  and  $(\Omega, \emptyset)^{c} = (\Omega^{c}, \emptyset^{c}) = (\emptyset, \Omega')$ . This means that the neutral

elements are complements of each other. In Set Theory  $\emptyset^c = \Omega$  and  $\Omega^c = \emptyset$  (exactly the same).

Definition 6. The symmetric-difference of two concepts  $(O_1, A_1)$  and  $(O_2, A_2)$  belonging to C is the concept  $(O_1 + O_2, (A_1 + A_2)^c) \in C$ , where  $O_1 + O_2$  and  $A_1 + A_2$  are the usual set-theoretic symmetric-

differences of  $O_1$  and  $O_2$  or  $A_1$  and  $A_2$ , respectively. In symbols:  $(O_1, A_1) + (O_2, A_2) = (O_1 + O_2, (A_1 + A_2)^c)$ .

The symmetric-difference of two concepts is well-defined because:

a.  $O_1 + O_2 \in \Omega$  and  $(A_1 + A_2)^c \in \Omega'$ . Therefore  $(O_1 + O_2, (A_1 + A_2)^c) \in C$ .

b. The operations symmetric-difference (+) and complement © are well-defined and therefore the results  $O_1 + O_2$  and  $(A_1 + A_2)^c$  are unique. Consequently the couple  $(O_1 + O_2, (A_1 + A_2)^c)$  is also unique.

Properties of the operation +

1. Commutative. Indeed,  $(O_1, A_1) + (O_2, A_2) = (O_1, + O_2, (A_1 + A_2)^c) =$ 

 $(O_2 + O_1, (A_2 + A_1)^c) =_{def.6} (O_2, A_2) + (O_1, A_1).$ 

 $^{*}$  because of the commutative property of the operation + .

2. Associative. Indeed,

 $(O_{1}, A_{1}) + [(O_{2}, A_{2}) + (O_{3}, A_{3})] = (O_{1}, A_{1}) + (O_{2}, O_{3}, A_{3})]_{def.6} (O_{1}, A_{1}) + (O_{2}, O_{3}, A_{3})]_{def.6} (O_{1}, A_{1}) + (O_{2}, O_{3}, A_{3})]_{def.6} = (O_{1} + (O_{2} + O_{3}), (A_{1} + (A_{2} + A_{3})))]_{aex} = (O_{1} + (O_{2} + O_{3}), (A_{1} + (A_{2} + A_{3}))]_{aex} = ((O_{1} + O_{2}) + O_{3}, ((A_{1} + A_{2}) + A_{3}))]_{aex} = ((O_{1} + O_{2}) + O_{3}, (A_{1} + A_{2})]_{aex} + A_{3})]_{aex} = ((O_{1} + O_{2}) + O_{3}, (A_{1} + A_{2})]_{aex} + A_{3})]_{aex} = ((O_{1} + O_{2}) + (O_{3}, A_{1} + A_{2})]_{aex} + (O_{3}, A_{3})]_{aex} = (O_{1} + O_{2}, (A_{1} + A_{2})]_{aex} + (O_{3}, A_{3})]_{aex} = [(O_{1}, A_{1}) + (O_{2}, A_{2})]_{aex} + (O_{3}, A_{3})]_{aex} = (O_{1}, A_{1}) + (O_{2}, A_{2})]_{aex} + (O_{3}, A_{3})]_{aex} = (O_{1}, A_{1}) + (O_{2}, A_{2})]_{aex} + (O_{3}, A_{3})]_{aex} = (O_{1}, A_{1}) + (O_{2}, A_{2})]_{aex} + (O_{3}, A_{3})]_{aex} = (O_{1}, A_{1}) + (O_{2}, A_{2})]_{aex} + (O_{3}, A_{3})]_{aex} = (O_{1}, A_{1}) + (O_{2}, A_{2})]_{aex} + (O_{3}, A_{3})]_{aex} = (O_{1}, A_{1}) + (O_{2}, A_{2})]_{aex} + (O_{1}, A_{1})]_{aex} + (O_{1}, A_{2})]_{aex} + (O_{1}, A_{3})]_{aex} + (O_{1}, A_{3})]_{$ 

\*\* because of the properties that hold for the usual symmetric-difference + between sets:

 $X + Y = X^{c} + Y = X + Y^{c} = X^{c} + Y^{c}$ \*\*\* because of the associative property between sets: X + (Y + Z) = (X + Y) + Z

3. We ask for neutral element.

(0, A) + (X, Y) = (X, Y) + (O, A) = (O, A), which, because of the commutative property of + : (property 1.), is reduced to the equation (O, A) + (X, Y) = (O, A)  $\Leftrightarrow_{def.6}$  (O + X, (A + Y)<sup>c</sup>) = (O, A)  $\Leftrightarrow_{def.6}$ : (O + X) = O and (A + Y)<sup>c</sup> = A)  $\Leftrightarrow_{****}$  (O + X = O and A + Y = A<sup>c</sup>)  $\Leftrightarrow_{*****}$  (X = O + O and Y = A<sup>c</sup> + A)  $\Leftrightarrow$  (X =  $\emptyset$  and Y =  $\Omega$ '). So, the neutral element of the operation + is the concept ( $\emptyset$ ,  $\Omega$ '), which, as

we know, is the neutral element of the operation  $\cup$  (in the same way as the set  $\emptyset$  is the neutral element for the operations  $\cup$  and +.

\*\*\*\* because of the property  $(X^c)^c = X$  i.e.  $(A + Y)^c = A \implies ((A + Y)^c)^c = A^c \implies$ 

$$\Rightarrow A + Y = A^c$$
.

\*\*\*\*\* because of the group structure of (P(X) enriched with the operation + (Where P(X) is the set of all subsets of a given set X). Since it is a group, fro every element (set) A of P(X) there exists its inverse or symmetric  $A^{L}$  so that their symmetric-difference gives the neutral element (in the same way that a + (-a) = 0, where a is a number). Consequently, we have:  $A + A^{L} = \emptyset$  (because  $\emptyset$  is the neutral element of the operation +). It is know that  $A^{L} = A$ ,  $\forall A \in P(X)$ . Therefore, in our case, we have:  $A + Y = A^{c} \Leftrightarrow (A + Y) + A = A^{c} + A \Leftrightarrow (A + A) + Y = \Omega' \Leftrightarrow \emptyset + Y = \Omega' \Leftrightarrow Y = \Omega'$ .

4. We ask for an inverse element.  $(0, A) + (X, Y) = (X, Y) + (0, A) = (\emptyset, \Omega') \underset{\text{prop.1}}{\Leftrightarrow} (0, A) + (X, Y) = (\emptyset, \Omega') \underset{\text{def.6}}{\Leftrightarrow} (0 + X, (A + Y)^{c}) = (\emptyset, \Omega') \Leftrightarrow (0 + X = \emptyset \text{ and } (A + Y)^{c} = \Omega') \Leftrightarrow (0 + X = \emptyset \text{ and } (A + Y)^{c} = \Omega') \Leftrightarrow (0 + X = \emptyset \text{ and } A + Y = \Omega'^{c}) \Leftrightarrow (0 + X = \emptyset) \text{ and}$  $A + Y = \emptyset) \underset{*****}{\Leftrightarrow} (X = 0 \text{ and } Y = A).$  So, the inverse element with reference to the operation + is the isometry in the second se

operation + and + respectively).

Conclusion

The set C, enriched with the operation +, is a group.

Moreover, we shall prove now that the set C, enriched with the operation + and  $\cap_{\bullet}$ , is a ring. Indeed, the two distributive properties hold.

5. Left-distributive property.  $(O_1, A_1) \cap (O_2, A_2) + (O_3, A_3)] \Leftrightarrow_{def.6} (O_1, A_1) \cap (O_2 + O_3, O_1) \otimes_{def.3} (O_1 - (O_2 + O_3), A_1 \cup (A_2 + A_3)^c) = O_1 \otimes_{******} = ((O_1 \cap O_2) + (O_1 \cap O_3), A_1 \cup (A_2 + A_3)^c) (1).$ 

At this point we make calculations only with the right part of the concept (1) that is with  $A_1 \cup (A_2 + A_3)^c$ . We have:

 $A_1 \cup (A_2 + A_3)C \underset{\bullet}{=} (A_1^C \cap A_2 + A_3))C \underset{*}{=}$ 

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 $= (A_{1}^{C} \cap (A_{2}^{C} + A_{3}^{C} \underset{******}{=}{=} (A_{1}^{C} \cap A_{2}^{C}) + (A_{1}^{C} \cap A_{3}^{C}))^{c}) =$   $= ((A_{1} \cup A_{2})C + (A_{1} \cup A_{3})C)C \underset{**}{=} ((A_{1} \cup A_{2}) + (A_{1} \cup A_{3}))C$ and hence the concept (1) becomes:  $((O_{1} \cap O_{2}) + (O_{1} \cap O_{3}), ((A_{1} \cup A_{2}) + (A_{1} \cup A_{3}))C)$   $= (A_{1} \cup A_{3})C) \underset{def.6}{=} (O_{1} \cap O_{2}, A_{1} \cup A_{2}) + (O_{1} \cap O_{3}, A_{1} \cup A_{3}) \underset{def.3}{=}$ 

 $[(O_1, A_1) \cap (O_2, A_2)] + (O_1, A_1) \cap [(O_3, A_3)]$  and so the proof is completed.

\*\*\*\*\*\*because of the distributive property of the usual intersection ( $\cap$ ) to the usual symmetric-difference (+).

 $\otimes$  because of the identity (X<sup>c</sup>)<sup>c</sup> = X and the Laws of de Morgan: (X  $\cup$  Y)<sup>c</sup> = X<sup>c</sup>  $\cap$  Y<sup>c</sup> and (X Y)<sup>c</sup> = X<sup>c</sup>  $\cup$  Y<sup>c</sup>.

6. Right-distributive property. That is:

 $[(O_2, A_2) + (O_3, A_3)] \cap (O_1, A_1) = [(O_2, A_2) \cap (O_1, A_1) + \\ \vdots \\ + [(O_3, A_3) \cap (O_1, A_1)].$  Obviously, this equality holds because of the commutative property of the : operation  $\cap$ .

Conclusion

The set C, enriched with the operation  $\bigcirc$  and +, is a ring. Moreover it is commutative (because of  $\vdots$  the commutative property of the operation  $\bigcirc$ , with unit (because of the neutral element  $\Omega, \emptyset$ ) of the operation  $\bigcirc$ ). The symmetric-difference + plays the role of addition and the intersection  $\bigcirc$  of  $\vdots$  multiplication.

Remarks

1. The operation  $\stackrel{\bullet}{\cup}$  is not distributive to the operation + (the same happens with  $\cup$  to +).

2. An element X is idempotent under the operation o if, and only if,  $X \circ X = X$ . In our case, every concept of C is idempotent under the operation  $\bigcirc$ . Indeed,

 $(0, A) \cap (0, A) =_{def.3} (0 \cap 0, A \cup A) = (0, A).$ 

The same happens with the set P(X) under the operation  $\bigcirc$ . Algebraic rings, like (P(X), +  $\bigcirc$ ) or (C, +,  $\bigcirc$ ), in which every element is idempotent, are called :

Boolean rings. So, (C, +,  $\bigcirc$ ) is a Boolean ring.

3. In Classic Set Theory, the symmetric-difference is defined as follows:  $X + Y = (X \cap Y^c) \cup (X^c \cup Y)$ .

We shall show that we could have given the analogous definition here and the result, after some calculations, is the formula of definition 6. Indeed, let's suppose that we had defined  $(O_1, A_1) + (O_2, A_2) + (O_2, A_3) + (O_3, A_3) + ($ 

 $\mathsf{A}_2) = [(\mathsf{O}_1, \mathsf{A}_1) \cap (\mathsf{O}_2, \mathsf{A}_2)^{\frown}] \cup$ 

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$$\begin{array}{l} \begin{array}{c} \begin{array}{c} \left( \left( 0_{1},A_{1} \right)^{C} & \left( 0_{2},A_{2} \right) \right) \\ \begin{array}{c} \left( \left( 0_{1}^{C},A_{1}^{C} \right) & \left( 0_{2},A_{2} \right) \right) \\ \end{array} \right) \\ \end{array} \\ \begin{array}{c} \left( \left( 0_{1}^{C},A_{1}^{C} \right) & \left( 0_{2},A_{2} \right) \right) \\ \end{array} \\ \begin{array}{c} \left( \left( 0_{1}^{C},A_{1}^{C} \right) & \left( 0_{2},A_{2} \right) \right) \\ \end{array} \\ \end{array} \\ \begin{array}{c} \left( \left( 0_{1}^{C},A_{1}^{C} \right) & \left( 0_{2},A_{2} \right) \right) \\ \end{array} \\ \begin{array}{c} \left( \left( 0_{1}^{C},A_{2}^{C} \right) & \left( 0_{1}^{C},A_{2}^{C} \right) \\ \end{array} \\ \end{array} \\ \begin{array}{c} \left( \left( 0_{1}^{C},A_{2}^{C} \right) & \left( 0_{1}^{C},A_{2}^{C} \right) \\ \end{array} \\ \begin{array}{c} \left( \left( 0_{1}^{C},A_{2}^{C} \right) & \left( 0_{1}^{C},A_{2}^{C} \right) \\ \end{array} \\ \begin{array}{c} \left( \left( A_{1}^{C},A_{2}^{C} \right) & \left( A_{1}^{C},A_{2}^{C} \right) \\ \end{array} \\ \end{array} \\ \begin{array}{c} \left( \left( A_{1}^{C},A_{2}^{C} \right) & \left( A_{1}^{C},A_{2}^{C} \right) \\ \end{array} \\ \end{array} \\ \begin{array}{c} \left( \left( A_{1}^{C},A_{2}^{C} \right) & \left( A_{1}^{C},A_{2}^{C} \right) \\ \end{array} \right) \\ \end{array} \\ \begin{array}{c} \left( \left( A_{1}^{C},A_{2}^{C} \right) & \left( A_{1}^{C},A_{2}^{C} \right) \\ \end{array} \right) \\ \end{array} \\ \begin{array}{c} \left( \left( A_{1}^{C},A_{2}^{C} \right) & \left( A_{1}^{C},A_{2}^{C} \right) \\ \end{array} \right) \\ \end{array} \\ \end{array} \\ \begin{array}{c} \left( \left( A_{1}^{C},A_{2}^{C} \right) & \left( A_{1}^{C},A_{2}^{C} \right) \\ \end{array} \right) \\ \end{array} \\ \end{array} \\ \begin{array}{c} \left( \left( A_{1}^{C},A_{2}^{C} \right) & \left( A_{1}^{C},A_{2}^{C} \right) \\ \end{array} \right) \\ \end{array} \\ \end{array} \\ \end{array} \\ \begin{array}{c} \left( \left( A_{1}^{C},A_{2}^{C} \right) & \left( A_{1}^{C},A_{2}^{C} \right) \\ \end{array} \right) \\ \end{array} \\ \end{array} \\ \end{array} \\ \end{array} \\ \begin{array}{c} \left( \left( A_{1}^{C},A_{2}^{C} \right) & \left( A_{1}^{C},A_{2}^{C} \right) \\ \end{array} \right) \\ \end{array} \\ \end{array} \\ \end{array}$$
 \\ \left. \left( \left( A\_{1}^{C},A\_{2}^{C} \right) & \left( A\_{1}^{C},A\_{2}^{C} \right) \\ \end{array} \right) \\ \end{array} \\ = \left( \left( A\_{1}^{C},A\_{2}^{C} \right) \\ \end{array} \right) \\ \left( \left( A\_{1}^{C},A\_{2}^{C} \right) \\ \end{array} \right) \\ = \left( \left( A\_{1}^{C},A\_{2}^{C} \right) \\ \end{array} \right) \\ \left( \left( A\_{1}^{C},A\_{2}^{C} \right) \\ = \left( \left( A\_{1}^{C},A\_{2}^{C} \right) \\ \end{array} \right) \\ \left( \left( A\_{1}^{C},A\_{2}^{C} \right) \\ \\ \left( A\_{1}^{C},A\_{2}^{C} \right) \\ \left( A\_{1}^{C},A\_{2}^{C} \right) \\ \\ \left( A\_{1}^{C},A\_{2}^{C} \right) \\ \left( A\_{1}^{C},A\_{2}^{C} \right) \\ \\ \left( A\_{1}^{C},A\_{2}^{C} \right) \\ \left( A\_{1}^{C},A\_{2}^{C} \right) \\ \\ \\ \left( A\_{1}^{C},A\_{2}^{C} \right) \\ \\ \left( A\_{1}^

That is, we found the formula of definition 6:  $(O_1, A_1) + (O_2, A_2) = (O_1 + O_2, A_$ 

 $(A_1 + A_2)^c$ ). Definition 6 is more easy for calculations and gives sense to the applications.

4. 
$$(0, a) + (\Omega, \emptyset) \underset{def.6}{=} (0, +\Omega, (A + \emptyset)^{c}) =$$
  

$$= (0^{c}, A^{c}) \underset{def.5}{=} (0, A)^{-\bullet} \Leftrightarrow (0, A)^{-\bullet} = (0, A) + (\Omega, \emptyset) \underset{*****}{\Leftrightarrow} \vdots$$

$$\Longrightarrow (0, A) + (0, A)^{-\bullet} = (0, A)^{-\bullet} [(0, A)^{-\bullet} (\Omega, \emptyset)] \underset{\text{prop.2}}{\Leftrightarrow} \vdots$$

$$\Leftrightarrow (0, A) + (0, A)^{-\bullet} = [(0, A) + (\Omega, \emptyset) \underset{*****}{\Leftrightarrow} \vdots$$

$$\Leftrightarrow (0, A) + (0, A)^{-\bullet} (\emptyset, \Omega') + (\Omega, \emptyset) \underset{def.6}{\Leftrightarrow} \vdots$$

$$\Leftrightarrow (0, A) + (0, A)^{-\bullet} = (\emptyset + \Omega, (\Omega' + \emptyset)^{c}) \Leftrightarrow$$

$$\vdots$$

$$\Leftrightarrow (0, A) + (0, A)^{-\bullet} = (\Omega, \emptyset) \text{ which is the neutral element of the operation } \cap.$$

The formula (0, A)  $^{\frown} = (0, A) + (\Omega, \emptyset)$ , though not simple as the definition 5, is easier in the : implementation using only one operation (and, especially, an operation of the ring structure). The formula (0, A) + (0, A)  $^{\frown} = (\Omega, \emptyset)$  is the analogous of thw property 1. of thw complement  $\subset \bullet$ :

where, instead of  $\,\cup$  , we have  $\,+\,.\,$ 

5. On the other hand, in the course of the above proof, we found:  $(0, A) + (0, A) = (\emptyset, \Omega')$  which is the neutral element of the operations  $\cup$  and +. Hence, only the : concept ( $(\emptyset, \Omega')$  in idempotent with reference to ( $\emptyset, \Omega'$ ). But (0, A)  $\stackrel{\bullet}{\cup}$  (0, A)  $\stackrel{=}{\underset{def.2}{\longrightarrow}}$  (0, A) and, hence,

every concept (O, A)  $\in$  C is idempotent with reference to  $\stackrel{\circ}{\cup}$ . Of course, we cannot speak about "the Boolean ring (C,  $\stackrel{\circ}{\cup}$ ,  $\stackrel{\circ}{\cap}$ )" because the set C, with the operation  $\stackrel{\circ}{\cup}$  instead of +, is not a ring.

6. In Classic Set Theory, two sets are called foreign (or incompatible in Probability Theory), when their intersection is the empty set. We try the same with concepts and  $\bigcirc$ :

 $(O_1, A_1) \cap (O_2, A_2) = _{def.3} (O_1 \cap O_2, A_1, \cup A_2) = (\emptyset, \Omega') \Leftrightarrow (O_1 \cap O_2 = \emptyset \text{ and } A_1 \cup A_2 = \Omega').$  That is, the sets of objects must be foreign but, also, must hold  $A_1 \cup A_2 = \Omega'.$ 

Two sets are called complementary if, and only if, their union is the maximum set of reference. We

try the same with concepts and  $\,\cup\!:\,$ 

 $(O_1, A_1) \cup (O_2, A_2) = (O_1 \cup O_2, A_1 \cap A_2) = (\Omega, \emptyset) \Leftrightarrow (O_1 \cup O_2) = \Omega \text{ and } A_1 \cap A_2 = \emptyset)$ . That is the sets of objects must be complementary and the sets of attributes foreign (the opposite results

from foreign concepts).

Having in mind the properties 1. and 2. of complement  $\subset \bullet$ , we see that a concept (O, A) and its complement (O, A)  $\subset \bullet$  are foreign and complementary.

For the definition of complementary sets we use their union ( $\cup$ ) and for concepts their union ( $\overline{\cup}$ ). Instead of the union ( $\cup$ ) we can use the symmetric-difference (+) when, and only when, their

intersection ( $\cap$ ) is the empty set. Instead of the union ( $\cup$ ) we can use the symmetric-difference (+) when, and only when, their intersection ( $\cap$ ) is the set ( $\emptyset$ ,  $\Omega$ ') or, equivalently, when they are :

foreign concepts. Since a concept and its complement are foreign concepts, we can use the symmetric-difference + instead of the union  $(\stackrel{\bullet}{\cup})$ . Besides, we know that from remark 4.

7. All the other identities know from Set Theory can be proved here using the concept-operations  $\cup$ ,  $\cap$ ,  $^{\circ}$  and + instead of the usual  $\cup$ ,  $\cap$ ,  $^{\circ}$  and +. For example, the very important Laws of De :

Morgan.

# **BOOLEAN ALGEBRA STRUCTURE**

(C, +,  $\cap$  is not a field.

Indeed, there should exist an inverse for the "multiplication"  $\bigcirc$ . Let's suppose that such an inverse (X, Y) does exist for an arbitrary concept (O, A)  $\in$  C. Equivalently,

 $(0, A) \cap (X, Y) = (X, Y) \cap (0, A) = (\Omega, \emptyset),$ 

where  $(\Omega, \emptyset)$  is the neutral element of the operation  $\cap_{\bullet}$ . Because of the commutative property of  $\cap_{\bullet}$  we have:

 $(\mathsf{O},\mathsf{A}) \, \mathop{\cap}_{\bullet}\, (\mathsf{X},\mathsf{Y}) = (\Omega, \varnothing) \, \underset{\mathrm{def},3}{\Leftrightarrow} \, (\mathsf{O} \, \cap \, \mathsf{X},\mathsf{A} \, \cup \, \mathsf{Y}) = (\Omega, \varnothing \, \Leftrightarrow)$ 

 $\Leftrightarrow (0 \cap X = \Omega \text{ and } A \cup Y = \emptyset) \Leftrightarrow$ 

 $\Leftrightarrow$  (0  $\cap$  X =  $\Omega$  and A  $\cup$  Y =  $\emptyset$ )  $\Leftrightarrow$ 

 $\Leftrightarrow$  (0 = X = Ω and A = Y = Ø). This means that only the concept (Ω, Ø) has inverse which is itself (Ω, Ø).

From Ring Theory, we remember of the ring with only one element which has the symbol (O). The unique element O plays every role in this structure which is a ring. The same happens with the element  $(\Omega, \emptyset) \in C$ .

(C, +,  $\bigcirc$ ) is not a field.

Indeed, there should exist an inverse for the "multiplication"  $\cap$ . Let's suppose that such an inverse

(X, Y) does exist for an arbitrary concept  $(O, A) \in C$ . Equivalently,

 $(\mathsf{O},\mathsf{A}) \cap (\mathsf{X},\mathsf{Y}) = (\mathsf{X},\mathsf{Y}) \cap (\mathsf{O},\mathsf{A}) = (\Omega,\varnothing),$ 

where  $(\Omega, \emptyset)$  is the neutral element of the operation  $\cap_{\bullet}$ . Because of the commutative property of  $\cap_{\bullet}$  we have:

(0, A)  $\cap_{def.3} (X, Y) = (\Omega, \emptyset) \Leftrightarrow_{def.3} (0 \cap X, A \cup Y) = (\Omega, \emptyset) \Leftrightarrow$   $\Leftrightarrow (0 \cap X = \Omega \text{ and } A \cup Y = \emptyset) \Leftrightarrow$  $\Leftrightarrow (0 = X = \Omega \text{ and } A = Y = \emptyset).$  This means that only the concept  $(\Omega, \emptyset)$  has inverse which is itself  $(\Omega, \emptyset)$ 

 $\emptyset$ ), From Ring Theory we remember of the ring with only one element which has the symbol (0). The unique element 0 plays every role in this structure which is a ring. The same happens with the element ( $\Omega, \emptyset$ )  $\in$  C.

(C, +,  $\cap$ ) is not a integral domain.

An integral domain is a commutative ring with unit (different from the neutral element of addition) which has no zero devisors. By zero divisor we mean an element of the ring, different from the neutral of addition, which, multiplied by another element of the ring, also different from the neutral element of addition, gives this neutral.

Let's suppose such a situation in C.  $(O_1, A_1) \cap (O_2, A_2) = (\emptyset, \Omega')$  where  $(O_1, A_1)$  and  $(O_2, A_2)$  are not

equal to  $(\emptyset, \Omega')$ . By definition 3. we take  $(O_1 \cap O_2, A_1 \cup A_2) = (O, \Omega') \Leftrightarrow (O_1 \cap O_2 = \emptyset \text{ and } A_1 \cup A_2 = \Omega')$  (2). If the concept  $(O_1, A_1)$  is given, then we can find at least one concept  $(O_2, A_2)$  satisfying the equations (2). Indeed, this is the complement  $(O_1, A_1) \stackrel{\sim}{\longrightarrow} \bigoplus_{d \in S} (O_1^C, A_1^C)$  of  $(O_1, A_1)$ . The two

concepts  $(O_1, A_1)$  and  $(O_2, A_2)$  must be foreign (because of equations (2)) and this is what happens with  $(O_1, A_1)$  and  $O_1, A_1) \stackrel{\frown}{\bullet}$  (as it was discussed in remark 6)\*. If  $(O_1, A_1)$  is given and different from the neutral  $(\emptyset, \Omega')$ , then it is obvious that the complement  $(O_1, A_1) \stackrel{\frown}{\bullet} = (O_1^C, A_1^C)$ ) is also different from  $(\emptyset, \Omega')^{**}$  and hence the concepts  $(O_1, A_1)$  and  $(O_1, A_1) \stackrel{\frown}{\bullet}$  are zero divisors. Consequently, C is not an integral domain.

\* The complement  $(O_1, A_1) \stackrel{\sim}{}$  is not the only concept foreign to  $(O_1, A_1)$  (in the same way that the foreign sets B of a given set A is not uniquely determined:  $A \cap \Omega B = \emptyset$  does not give us one and unique B. Besides,  $(P(X), \Omega)$  is not a group and so the equation  $A \cap B = \emptyset$  does not have a unique solution). We use  $(O_1, A_1) \stackrel{\sim}{}$  as an easy and known example.

\*\* In case  $(O_1, A_1) = (\Omega, \emptyset)$ , which is really different from the neutral  $(\emptyset, \Omega')$ , its complement is the couple  $(\emptyset, \Omega')$  and hence the two concepts are not an example of zero divisors, since they should be both different from the neutral  $(\emptyset, \Omega')$ . This case does not prevent us from characterizing C as non-integral domain. Even if we find only two zero divisors, we say that the ring is not an integral domain.

# A property of the Boolean rings

Boolean ring is called a ring where all its elements are idempotent. As we know from remark 2, (C, +,  $\cap$ ) is a Boolean ring.

•

From idempotence we have the following results:

1. C is a commutative ring (which we have proved independently of the idepotence).

2. For every  $(0, A) \in C$ , its additive inverse, that is, its inverse because of the group structure (C,

+), is its own-self. Equivalently,  $(0, A) + (0, A) = (\emptyset, \Omega')$ , as we have seen in remarks 4 and 5. :

This is a very characteristic result of Boolean rings but we must not forget that it comes from the idempotence.

3. If there are at least three elements in C, then C is not an integral domain (which we have already proved).

What is new here and very interesting is the case that C has only two elements. Since C is, in every case, an (idempotent) ring, it follows that the one element is the neutral element of the addition +,

that is the couple ( $\emptyset, \Omega'$ ). On the other hand, since we want to prove C to be an integral domain, there must exist the unit in C (that is the neutral element of the multiplication  $\cap$ ). Hence, the second

element, is the couple  $(\Omega, \emptyset)$ . The set  $\{(\emptyset, \Omega'), (\Omega, \emptyset)\} = C$ , enriched with the operations + and  $\cap$ , is a Boolean ring

(idempotent for both elements, commutative), it has the unit for multiplication  $(\Omega, \emptyset) \neq (\emptyset, \Omega')$  and it is an integral domain. Indeed, for the "non-zero" element  $(\Omega, \emptyset)$ , there is no other "non-zero" element so that their multiplication gives the neutral element  $(\Omega, \emptyset)$ . Hence,  $(\Omega, \emptyset)$  is not a zero divisor and so C is an integral domain.

Moreover, C is a field - this is the unique case of field. This happens because the only "non-zero" element  $(\Omega, \emptyset)$  is the inverse of itself with reference to the multiplication  $\cap$ . It is exactly the famous

field {0, 1}. This exception from integral domains with two elements, was partly indicated in the proof that C is not an integral domain and especially in the remark \*\*.

(C, +,  $\cap$ ) is a Boolean algebra.

:

Indeed, Boolean algebra is a Boolean ring with unit. We know that the unit in C is the couple  $(\Omega, \emptyset)$ . In the Theory of Lattices, a Boolean algebra is a distributive and complementary lattice.

We have already proved that C is a distributive lattice. For the characterization of complementary we must prove the following three properties:

1. The neutral element ( $\emptyset$ ,  $\Omega$ ') belongs to C and it is valid that:

 $(\emptyset, \Omega') \underline{\subset} (0, A), \forall (0, A) \in C$ 

2. The neutral element  $(\Omega, \emptyset)$  belongs to C and it is valid that:

 $(\Omega, \varnothing) \bullet \supset (0, A), \forall (0, A) \in C.$ 3.  $\forall (0, A) \in C \exists (0, A)^{\circ} \in C:$  $(0, A) \cap (0, A)^{\circ} = (\varnothing, \Omega')$  and

•

 $(\mathsf{O},\mathsf{A})$   $\cup$   $(\mathsf{O},\mathsf{A})^\circ = (\emptyset,\Omega').$ 

The three properties above are valid in C. Proof:

1. We know that  $(\emptyset, \Omega') \in C$  and  $(\emptyset, \Omega') \subseteq \bullet (0, A) \underset{\underline{def.4a}}{\Leftrightarrow} (\emptyset \subseteq 0 \text{ and } \Omega' \supseteq A)$  really hold  $\forall (o, A) \in C$ 

C.

2. We know that  $(\emptyset, \Omega') \in \mathbb{C}$  and  $(\emptyset, \Omega') \xrightarrow{\bullet \supset} (0, A) \Leftrightarrow (0, A) \xrightarrow{\frown \bullet} (\Omega, \emptyset) \Leftrightarrow_{def.4a} (0 \xrightarrow{\frown} \Omega \text{ and } A \xrightarrow{\frown} \emptyset)$ 

which really hold  $\forall$  (O, A)  $\in$  C.

3. Obviously, (O, A) $^{\circ}$  is the complement (O, A)  $^{-\bullet}$  .

Equation (0, A)  $\cap$  (0, A)<sup>o</sup> = ( $\emptyset$ ,  $\Omega$ ') shows us that (0, A) and (0, A)<sup>o</sup> are foreign to each other and,

hence, with given (O, A), (O, A)  $^\circ$  is not uniquely determined.

Equation  $(0, A) \cup (0, A)^{\circ} = (\Omega, \emptyset)$  shows us that (0, A) and  $(0, A)^{\circ}$  are complementary to each other and, hence, with given (0, A),  $(0, A)^{\circ}$  is not uniquely determined (IB3 remark 6).

However, both equations determine uniquely as a solution (with given (0, A)) the complement (0,A)  $^{-\bullet}$ . Uniqueness of the complement is a characteristic property of the Boolean algebras. There are complementary lattices but this is not enough. Only when the complementary lattice is also distributive (and hence it becomes Boolean algebra), only then the complement is uniquely determined and three other important properties are valid:

1.  $((0, A)^{-\bullet})^{-\bullet} = (0, A), \forall (0, A) \in C.$ 

2. The two laws of de Morgan:

a. 
$$[(O_1, A_1) \cap (O_2, A_2)] \stackrel{\frown}{=} (O_1, A_1) \stackrel{\frown}{\cup} (O_2, A_2) \stackrel{\frown}{=}$$

b. 
$$[(O_1, A_1) \stackrel{\bullet}{\cup} (O_2, A_2)] \stackrel{\frown}{=} = (O_1, A_1) \stackrel{\frown}{-} \cap (O_2, A_2) \stackrel{\frown}{=}$$

3.  $(O_1, A_1) \underbrace{{}_{\scriptstyle{\frown}} \bullet} (O_2, A_2) \Longrightarrow (O_2, A_2) \overset{{}_{\scriptstyle{\frown}} \bullet} \underbrace{{}_{\scriptstyle{\bullet} \supset}} (O_1, A_1) \overset{{}_{\scriptstyle{\frown}} \bullet}$ 

# Lattices, algebraic rings and Boolean algebras

A lattice L has an order relation  $\lambda$ . With the help of  $\lambda$  we define the supremum and infimum of two element of L, which are two operations in L with the properties commutative, associative, absorbent and idempotent (procedure one).

On the other hand, a set L enriched with two operations v and  $\Lambda$  that have the above properties, is proved to be a lattice and the order relation  $\leq$  is given by the equivalences:

$$X \leq \Psi \Leftrightarrow_{\text{def.}} X \lor \Psi = \Psi \Leftrightarrow X \land \Psi = X \text{ (procedure two).}$$

This is the case of definition 4a: we have the operations  $\,\cup\,$  and  $\,\cap\,$  with the above properties and

from these operations we defined the order relation  $\ \ \subset \bullet$ .

Afterwards, having now an order relation  $\leq$ , we define the supremum and infimum. Then, it is proved that the supremum and infimum are given by the two pre-existing operations v and  $\Lambda$ . Precisely:

sup  $\{X, \Psi\} = X \vee \Psi$  and  $\inf \{X, \Psi\} = X \wedge \Psi$ 

Indeed, it is easily proved that the supremum of two sets A and B is their union A  $\,\cup\,$  B and the

infimum is their intersection  $A \cap B$ . Or, in the case of C, sup {  $(O_1, A_1)$ ,  $(O_2, A_2)$  } =  $(O_1, A_1)$ ,  $\bigcup$   $(O_2, A_2)$  and inf {  $(O_1, A_1)$ ,  $(O_2, A_2)$  } =  $(O_1, A_1) \cap (O_2, A_2)$ .

A very important result (not proved here) is the following.

We begin with a lattice (L,  $\lambda$  ) and, using the supremum and infimum, we give to it an algebraic structure (by procedure one) .

Then, since now we have two operations, by procedure two we define an order relation  $\leq$  which is proved to be the same with the order  $\lambda$  from which we started.

Similarly, we can start from an algebraic structure and, by procedure two, we define an order relation  $\leq$ . Then, by procedure one, we define two operations (the supremum and infimum) and so we have again an algebraic structure. It is proved that the operations v and  $\Lambda$  from which we started are the same with the new operations supremum and infimum.

The practical value of this result for us is that the order structure and the algebraic structure of a lattice are equivalent (under some conditions), independently of the kind of order relation or of the operations.

A final matter is the role of the operations  $\cup$  and +. A Boolean algebra is a lattice with the

operations  $\cup$  and  $\cap$ . It is a Boolean ring with the unit ( $\Omega, \emptyset$ ) of  $\cap$ . Only with  $\cup$  and  $\cap$  it is not an

algebraic ring. To make it algebraic ring we need the addition + which makes it a group while  $\dot{\cup}$ 

does not make it a group. The multiplication of the algebraic ring is the operation  $\cap_{\bullet}$  of the Boolean ring. Precisely:

a. The Boolean algebra (C,  $\cup$ ,  $\bigcirc$ ) becomes the algebraic ring (C, +,  $\bigcirc$ ) with the formulas:

$$((O_1, A_1) + (O_2, A_2) \stackrel{=}{_{def.}} ((O_1, A_1) \cap (O_2, A_2))^{\frown} \cup ((O_1, A_1)^{\frown} \cap (O_2, A_2) \text{ and } (O_1, A_1) \cap (O_2, A_2)$$

the same (the complement  $\sub{} \bullet$  comes from the Boolean algebra structure).

b. The algebraic ring (C, +,  $\bigcirc$ ) becomes the Boolean algebra (C,  $\bigcirc$ ,  $\bigcirc$ ) with the formulas:

 $(O_{1}, A_{1}) \cap (O_{2}, A_{2}) = (O_{1} \cap O_{2}, A_{1}, \cup A_{2}) \text{ and}$   $(O_{1}, A_{1}) \cup (O_{2}, A_{2}) \stackrel{=}{_{def.}} (O_{1}, A_{1}) + (O_{2}, A_{2}) + (O_{1}, A_{1}) \cap (O_{2}, A_{2})$  $\vdots$ 

In the above formulas, if we make the calculations we shall find the original definitions of  $\cup$  and + .

(definitions 2. and 6.).

If we start from an algebraic ring and make it Boolean algebra and then take this Boolean algebra and make it an algebraic ring, we shall find the operations of the algebraic ring from which we started. The same happens if we start from a Boolean algebra.

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