Based Substitution

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#### Abstract

In order to maximize expected profit in a single period setting we deem the problem of a individualist retailer for a discrete set of products with varying prices and costs choosing the optimal assortment and inventory levels. Let us assume 'type', which denotes the ranking of the products by the order of the preferences. Dynamic substitution takes place when a customer at the time of their visit buys the highest ranked product available (if any) from the assortment. Considering the first case, when all the customers are of the same consumer type and we show that it may be optimal to stock multiple products because of differences in their round about risk-return trade-offs. With the help of dynamic programming algorithm, the solution is obtained. When customers are partitioned into different types in fixed proportions, we will show that in this case algorithm gives the optimal solution. When the number of customers of each type is random, we use the algorithm to construct two heuristics and an upper bound, also numerically evaluate the performance of the heuristics.


## INTRODUCTION

It was observed that the problem under dynamic substitution (representative of real-life applications) consider to be difficult to solve than the static substitution. Mahajan and Van [1], proposed a sample path gradient based heuristic to solve this problem. Second heuristic for this problem was proposed by Smith and Agrawal [2] and since it depends upon static substitution therefore contain calculations in the simpler form. Gaur and Honhon [3] proposed the third heuristic that based on retailer-controlled substitution in order to maximize the profits.
Our intension is to scrutinize the course group of consumer choice models for which the problems under the dynamic substitution dealing the both cases assortment planning and inventory management can be solved to the optimality. Also we show that the optimal solution for the different type of preference structures is obtained competently even though the profit function is not quasiconcave in inventory levels.
When all the customers are of same tastes gives the simplest preference structure, in this a single customer type is denoted by $(1,2, \ldots . . . . . ., n)$, where $n$ is the number of available products. With the help of proficient dynamic algorithm having complexity $O\left(n^{2}\right)$, we locate the optimal set of products to stock and its corresponding vector for the given problem in a single-period setting. It was observed that the retailer may do well by stocking a single product. Certainly, the optimal static substitution solution consists of stocking one most profitable product. On the other hand we show that the optimal solution may offer more than one product, and that the value function is nondifferentiable in the inventories of products. Then we consider a preference model called nested preferences, in which a customer of type ( $1, \ldots . .$. , i) will prefer the products 1 through $i$ and in that order but will not buy product $\mathrm{i}+1$ if products 1 to i are not available. We show that the same algorithm can be used to solve the assortment problem to optimality when all customers are of the same type but the retailer does not know a priori which one it is, and assigns probability $\delta_{i}$ to type $(1, \ldots . .$, , $)$ (this is labelled as a trend-following population). The algorithm also applies to the case where there is a fixed proportion $\delta_{i}$ of customers of each type (fixed proportion).
This solution is then shown to provide an upper bound and a heuristic for the case in which each customer can independently be of each type with probability $\delta_{i}$ (random proportion). Finally, we
consider the case where the set of consumer types can be represented by an outtree, and there is a fixed proportion of customers of each type and then show that the algorithm provides an optimal solution in this case as well. For the random proportion case, we numerically test the performance of a modified version and that of the fixed proportion heuristic. Using two previously known heuristics, one based on static substitution, and the other, the Sample Path Gradient Algorithm of Mahajan and van Ryzin [1], we yardstick the results. Numerical studies involves two heuristics yield average optimality gaps of $0.14 \%$ and $0.18 \%$, while the static substitution-based heuristic yields an average optimality gap of $1.57 \%$, and the sample path gradient algorithm yields $0.54 \%$. In our heuristics, the optimality gaps decrease as mean demand increases and as the proportion of customers willing to substitute increases. Comparison with the static substitution heuristic reveals that the profit loss caused by ignoring substitution due to stock-outs can be substantial. Undeniably, we theoretically show that there are conditions in which a static-substitution based heuristic can be arbitrarily far from the optimal solution, and thus, dynamic substitution plays a significant role in determining profits.
The optimality gap of our algorithm is similar to or better than that of the SPGA in almost all problem instances, and further, significantly improves computation time. However, our algorithm is restricted to the specific nature of consumer choice while the SPGA is very general. A significant outcome of our analysis is that even when all consumers have identical preferences, it is optimal for a retailer to offer more than one product in the optimal assortment. This result shows that dissimilar costs and prices of products provide a rationale for variety under dynamic substitution. With regard to heterogeneous tastes, economists have explained the degree of variety in a given product category as the result of the interplay between the demand for variety from consumers and the cost of providing the variety. Van Ryzin \& Mahajan [4] and, Gaur \& Honhon 3[2] shows that heterogeneity and uncertain preferences of the customer increases the variety whereas inventory costs limits the variety when customers choose according to the multinomial logit choice model or a locational choice model, respectively. Cachon et al. [5] show that multi-product oligopolistic competition, lower search costs lead to larger assortments. In our preference model, the traditional reasons for offering variety are assumed away because the retailer is a monopolist and the consumer population has homogeneous tastes and vague undeniable preferences.
Consequently we show that varying inventory cost economics of products constitute so far another motive for offering variety when customers substitute dynamically. Products with varying inventory cost economics have different risk-return profiles, and thus, variety becomes a mechanism to manage the risk due to demand uncertainty. Imminent to all, the algorithm we use to solve the assortment planning problem allows us to obtain structural imminent into the optimal solution. Predominantly, we show that the optimal assortment does not necessarily contain the most profitable or the most preferred product.
Thus we obtain the first set of optimal results on assortment planning with dynamic substitution. This is done by using an efficient algorithm which can also be used to provide a heuristic for a more general preference structure and we also provide an upper bound on the optimal expected profit. Finally we obtain new insights on the product variety question, since the retailer can in some cases manage demand risk better with a larger assortment. Van Ryzin and Mahajan (41999) were the first to study assortment planning and inventory decisions under the MNL model for the case of static substitution with exogenous prices to determine many properties of the optimal solution, in which the optimal assortment consists of the most popular products from the finite set of potential products to offer.
Aydin and Ryan [6] use the MNL model to study the both, assortment planning and pricing problem which comes under static substitution. They conclude that the optimal solution is such that all products have equal difference between price and cost. Cachon et al. [5] show that ignoring consumer search in demand estimation can result in an assortment with less variety and significantly lower expected profits compared to the optimal solution. The search costs can induce a retailer to carry an unprofitable product in its assortment to reduce consumer search. Kok and Fisher [7] estimate assortment-based substitution in an MNL model by leveraging data from stores with varying assortments and present an algorithm to solve the assortment planning problem with one-level stock out based substitution in the presence of shelf-space constraints. De Groote [8] and Alptekinoglu \& Corbett [9] assimilate product differentiation and inventory costs in the context of

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locational choice models with uniform deterministic demand and customer preferences. Alptekinoglu \& Corbett explore competitive positioning and pricing for two firms in which one offering infinite variety through mass customization and the other one offering a finite set of different products.
They show the counterintuitive result that the mass producer needs to reduce variety to soften price competition with the mass customizer firm where as De Groote mull over a monopoly firm and explore the coordination between the marketing decision of product line breadth and the operations decision of production flexibility. Chen et al. [10] cram the optimal product positioning and pricing, extending Lancaster's model to slot in varying prices and quality levels in the attribute space, as well as varying reservation prices of customers also the optimal solution for this model under stochastic demand and static substitution can be constructed using dynamic programming by utilizing a `crosspoint property' to determine the demands for individual products.
Lastly Gaur \& Honhon [4] unite inventory management problem with the the optimal assortment problem to obtain the optimal solution under static substitution for horizontally differentiated products in the context of a one-dimensional locational choice model. Also they pioneer a stochastic of demand and a unimodal distribution of customers on the attribute space and state that the optimal assortment is such that products should be equally spaced and that it is not necessary optimal to stock the product located at the mode of the distribution. Now in the mentioned way this research is related to our paper:
(i). Getting rid of the dependency of demand on inventory levels, it is possible to solve the problem of assortment planning under static substitution predates the research under dynamic substitution optimally, though the assumption of static substitution has limited applications in real-life settings.
(ii). Through this research a rich set of consumer choice models may be considered under the dynamic substitution and in the end it shows the practical relevance of the assortment planning and inventory management problem.
Section 2 illustrates the consumer choice model and profit assumptions.
Section 3 involves the solution to the case where all customers have similar preferences. Sections 4 and 5 extend the results to heterogeneous choice models.
Section 6 containing the, Numerical results.
All proofs are in the Appendix, unless otherwise stated.

## MODEL

### 1.1 Consumer Choice Model

Mull over a product category consisting of $m$ potential products to stock, indexed from 1 to $r$.
Consumer type is defined as a vector of products that a customer would be willing to buy in decreasing order of preference. For example, a customer of type $(1,2,3)$ prefers to buy product 1 if it is available, product 2 if product 1 is not available, product 3 if products 1 and 2 are not available and nothing otherwise. In general, a type $u$ is a vector ( $u_{1}, \ldots . . . . . .$. , us) such that $\mathrm{s} \leq r, u_{\alpha} \in$ $\{1, \ldots \ldots \ldots ., r\}$ for $\alpha=1, \ldots \ldots \ldots, s$ and $u_{1} \neq u_{2} \neq \cdots \cdots \neq u_{s}$. The number of possible consumer types in a product category with r variants can be as large as $\sum_{k=0}^{r} \frac{r!}{(r-k)!}$. Though, some of these types do not have any practical sagacity. That is, it is found that a product like refined, customers who prefer a Suffola refined are not likely to switch to a Sunflower refined or Nature refined as their second choice. Therefore we define a preference structure as a set of restrictions on the possible consumer types. Let $U$ be the set of all consumer types that satisfy those restrictions. We also define a partitioning mechanism of demand as a method to specify how the customers are split between each possible type in U . We refer to the combination of the preference structure and the partitioning mechanism as a consumer choice model. The consumer choice model is a homogeneous population model when there is only one consumer type whereas it is a heterogeneous population model when there is more than one consumer type.
Let us deemed in our paper preference structures which are ordered from the most to the less restrictive in the following way
a) There is only one possible consumer type which, without loss of generality, can be defined as $(1, \ldots \ldots ., r)$ is known as Homogeneous population.
b) The set of possible consumer types can be defined without loss of generality as
$U=\{(1),(1,2), \cdots \cdots \cdots,(1,2, \cdots \cdots \cdots, r)\}$ known as Nested preferences.

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c) When the set of possible consumer types can be represented by an outtree, that is a directed graph in which nodes represent products, there is a single initial note representing the first choice product for all consumer types and there is a unique directed path from the initial node to any other node known as Outtree-shaped preferences.
In our paper we deemed the following partitioning mechanisms for the heterogeneous consumer choice models:
a) When customers are of the same consumer type but the retailer does not know a priori which type it is and assigns a probability to each possible type, they comes under Trend-following partitioning or herd-behavior.
b) When customers are split between the different consumer types in fixed proportions, comes under Fixed partitioning.
c) If the number of customers of each type is a random variable with known distribution comes under Random partitioning. If we assume that each customer has a given probability of being of each possible type, then the number of customers of each type follows a multinomial distribution.
Make a note that the type of a customer can be the result of a utility maximization process, that is, each customer assigns a utility $R_{i}$ to product $\mathrm{i}=1, \ldots . ., \mathrm{n}$ and sets the utility of not purchasing any product to $\mathrm{R}_{0}$. Let $\mathrm{R}[\mathrm{j}]$ be the jth greatest value of the utility $\operatorname{vector}\left(R_{0}, R_{1}, \ldots \ldots R_{r}\right)$, the type of the customer is $\left(u_{1}, u_{2}, \ldots \ldots, u_{m}\right)$ if $R_{u_{j}}=R_{[j]}$ for $\mathrm{j}=1, \ldots, \mathrm{~s}$ and $R_{0}=R_{[s+1]}$. It is observed that a retailer does not need to know the utilities in order to use this model, they only needs to estimate the set of possible consumer types for the given product category and associate a probability to each one of them.
Considering that customer preferences are not affected by the retailer decisions. Meticulously, the prices of the products are taken to be exogenous variables in this model.

### 1.2 Demand

Let us consider the following:

| $x_{k}$ | Inventory level of the product $k$ |
| :--- | :--- |
| $x=\left(x_{1}, \ldots ., x_{r}\right)$ | The inventory vector |
| P | The random variable denoting total number of customers coming to the <br> store with cdf F, pdf f and mean $\mu$ |
| $\mathrm{V} \leq \mathrm{x}$ | The inventory vector seen by a customer of type $u=\left(u_{1}, \ldots ., u_{s}\right)$. |
| $A_{k}$ | Demand for product k (defined as the number of customers who attempt <br> to buy product k |
| $s_{k}$ | Selling price of the product k |
| $p_{k}$ | Purchasing cost of the product k |
| $n_{k}$ | Savage value for $\mathrm{k}=1, \ldots . . \mathrm{r}$ |
| $c_{k}$ | every time a customer attempts to buy product k but does not find it, the <br> retailer incurs a penalty cost of this product |
| $u_{k}=s_{k}+\sum_{i=1}^{k} c_{i}-p_{k}$. | Underage cost of product k |
| $o_{k}=p_{k}-n_{k}$ | Overage cost of product k |

In the dynamic substitution, each customer buys the highest ranked product available (if any) in the assortment at the time of their visit to the store. So the customer buys product $u_{q} \in u$ iff
$x_{u_{1}}=\ldots \ldots .=x_{u_{q-1}}$. Customer goes to buy products $u_{1}$ to $u_{q}$ in this case. Since each customer may be counted multiple times therefore we have $\sum_{k=1}^{r} A_{k} \geq A$.
$\operatorname{Cost} c_{k}$ measures the loss of goodwill associated with the customer not buying their first choice and having to substitute to a less preferred product, possibly incurring a search cost. The single-period (newsvendor) expected profit for the product category given by,
$P_{E} \Pi(x)=\sum_{k=1}^{r}\left\{s_{k} P_{E}\left[\min \left(x_{k}, A_{k}\right)\right]-p_{k} x_{k}+n_{k} P_{E}\left[x_{k}-A_{k}\right]^{+}-c_{k} P_{E}\left[A_{k}-x_{k}\right]^{+}\right\}$.

For the product category, the one-period (newsvendor) expected profit given is by

$$
\begin{aligned}
P_{E} \Pi(x)=\sum_{k=1}^{r}\left\{\left(s_{k}+\sum_{i=1}^{k} c_{i}\right.\right. & \left.\left.-p_{k}\right) x_{k}-\left(s_{k}+\sum_{i=1}^{k} c_{i}-n_{k}\right) P_{E}\left[x_{k}-A_{k}\right]+\right\}-\lambda \sum_{k=1}^{r} c_{k} . \\
& =\sum_{k-1}^{r}\left\{u_{k} x_{k}-\left(u_{k}+o_{k}\right) P_{E}\left[x_{k}-A_{k}\right]^{+}\right\}-\lambda \sum_{k=1}^{r} c_{k}
\end{aligned}
$$

For our ease we consider that $c_{k}=0$ for $k=1, \ldots \ldots, r$.
The goal of the retailer is to find $x^{*}$ such that
$P_{E} \Pi\left(x^{*}\right)=\max _{x \geq 0} P_{E} \Pi(x)$.

## The Homogeneous Population model

The homogeneous population model consists of all customers of the same consumer type,
that is $(1,2, \ldots, r)$. It can be described in way that all customers prefer to buy product 1 if it is available, otherwise they prefer to buy product 2 if it is available, and so on, up to product r. And they do not buy anything if and only if products 1 to $r$ are not available.
The demand for product $\mathrm{k}, A_{k}^{p}$, where p is a Homogeneous preferences, depends on the inventory levels of product 1 to $\mathrm{k}-1$, and is given by:

$$
A_{k}^{h}=A
$$

$A_{k}^{p}\left(x_{1}, \ldots ., x_{k-1} 1\right)=\left[A-\sum_{j=1}^{k-1} x_{j}\right]^{+}$for $k=2, \ldots . ., r$.
for the product category the one-period (newsvendor) expected profit is given by:

$$
\begin{align*}
& \quad P_{E} \prod(x)=\sum_{k=k}^{r}\left\{u_{k} x_{k}-\left(u_{k}+o_{k}\right) P_{E}\left[q_{k}-A_{k}^{p}\right]+\right\} \\
& =\sum_{k=1}^{r}\left[u_{k} q_{k}-\left(u_{k}+o_{k}\right)\left(x_{k} G\left(\sum_{j=1}^{k-1} x_{j}\right)+\int_{\sum_{j=1}^{k=1}=1}^{\sum_{j}^{k} x_{j}} x_{j}\left(\sum_{j=1}^{k} x_{j}-v\right) g(v) d v\right)\right] \tag{1}
\end{align*}
$$

We will show that the retailer would stock only the most preferred one, if we considering the case when no two products have the same overage and underage costs.

### 2.1 Two-product problem

A retailer may offer more than one product in their assortment due to the differences in the price and cost parameters of products. Also cost economics can be a driver of product variety when customers dynamically substitute in the event of a stock-out, even in the absence of competitive pressures and customer heterogeneity. In our model the tradeoffs and computational problems can be demonstrated by using a two-product problem for which the expected profit for $\mathrm{n}=2$ is given by:

$$
\begin{align*}
& P_{E} \Pi^{p}\left(x_{1}, x_{2}\right)=u_{1} x_{1}+\left(u_{1}+o_{1}\right) \int_{0}^{x}(q-v) f(v) d v+u_{2} x_{2} \\
&-\left(u_{2}+o_{2}\right)\left(x_{2} G\left(x_{1}\right)+\int_{x_{1}}^{x_{1}+x_{2}}\left(x_{1}+x_{2}-v\right) g(v) d v\right) \tag{2}
\end{align*}
$$

First derivative of expected profit with respect to $x_{2}$ is given by:

$$
\frac{\partial P_{E} \Pi^{p}(x)}{\partial x_{2}}=u_{2}-\left(u_{2}+o_{2}\right) G\left(x_{1}+x_{2}\right)
$$

Given that $P_{E} \Pi^{p}$ is concave in $x_{2}$, the optimal value of $x_{2}$, as a function of $x_{1}$ is equal to

$$
\begin{equation*}
x_{2}=\left[G^{-1}\left(\emptyset_{2}\right)-x_{1}\right]^{+} \text {where } \emptyset_{2}=\frac{u_{2}}{u_{2}+o_{2}} . \tag{3}
\end{equation*}
$$

Substituting this value in eqn. (2), we get:

$$
\begin{align*}
P_{E} \Pi^{p}\left(x_{1}\right)= & \left(u_{1}-u_{2}\right) x_{1}-\left[\left(u_{1}+o_{1}\right)-\left(u_{2}+o_{2}\right)\right] \int_{0}^{x_{1}}\left(x_{1}-v\right) g(v) d v+u_{2} G^{-1}\left(\emptyset_{2}\right)- \\
& \left(u_{2}+o_{2}\right) \int_{0}^{G^{-1}\left(\phi_{2}\right)}\left(G^{-1}\left(x_{2}\right)-v\right) g(v) d v  \tag{4}\\
& \text { if } x_{1}<G^{-1}\left(\emptyset_{2}\right) \\
& =u_{1} x_{1}-\left(u_{1}+o_{1}\right) \int_{0}\left(x_{1}-v\right) g(v) d v
\end{align*} \text { if } x_{1} \geq G^{-1}\left(\emptyset_{2}\right) .
$$

The derivative of $P_{E} \Pi^{p}\left(x_{1}\right)$ is continuous with respect to $x_{1}$. That is:

$$
\begin{array}{cr}
\frac{\partial P_{E} \Pi^{p}\left(x_{1}\right)}{\partial x_{1}}=\left(u_{1}-u_{2}\right)-\left[\left(u_{1}+o_{1}\right)-\left(u_{2}+o_{2}\right)\right] G\left(x_{1}\right) & \text { if } x_{1}<G^{-1}\left(\emptyset_{2}\right)  \tag{5}\\
=u_{1}-\left(u_{1}+o_{1}\right) G\left(x_{1}\right) & \text { if } x_{1} \geq G^{-1}\left(\emptyset_{2}\right) \\
u_{1}-u_{2} &
\end{array}
$$

Case 1
Stock both
$x_{1}=G^{-1}\left(\emptyset_{12}\right), x_{2}=G^{-1}\left(\emptyset_{2}\right)$

Case 2
Stock only product 1

$$
x_{1}=G^{-1}\left(\emptyset_{1}\right), x_{2}=0
$$

Case 1
Stock only product 2 $x_{1}=0, x_{2}=G^{-1}\left(\emptyset_{2}\right)$

Case 2
Stock only product 1or only product 2
$x_{1}=0, x_{2}=G^{-1}\left(\emptyset_{2}\right)$ or $x_{1}=G^{-1}\left(\emptyset_{1}\right), x_{2}=0$

Figure 1: 2-product problem solution
Two things are to be noted:

1. The expected profit function is not necessarily concave even for a basic problem with homogeneous consumer but it hardly matters and $P_{E} \Pi^{p}$ is not concave in $x_{1}$ if $u_{1}+o_{1}<$ $u_{2}+o_{2}$.
2. The global maxima of $P_{E} \Pi^{p}\left(x_{1}\right)$ can occur at $x_{1}=0$ or at a point where the first order conditions are satisfied. Two functions in eqn. (5) have unique stationary points equal to
$G^{-1}\left(\emptyset_{12}\right)\left[\right.$ where $\left.\emptyset_{12}=\frac{u_{1}-u_{2}}{\left(u_{1}+o_{1}\right)+\left(u_{2}+o_{2}\right)}\right]$ and $G^{-1}\left(\emptyset_{1}\right)\left[\right.$ where $\left.\emptyset_{1}=\frac{u_{1}}{u_{1}+o_{1}}\right]$ respectively.
Since $P_{E} \Pi^{p}\left(x_{1}\right)$ is differentiable everywhere, it can have at most one interior local maximum, either at $G^{-1}\left(\emptyset_{12}\right)$ or at $G^{-1}\left(\emptyset_{1}\right)$.
On the signs of $u_{1}-u_{2}$ and $\emptyset_{1}-\emptyset_{2}$, the optimal solution depends. Keeping in mind the concept that if $u_{1} \leq u_{2}$, then $P_{E} \Pi^{\mathrm{p}}\left(\mathrm{x}_{1}\right)$ is decreasing at zero, and therefore, $x_{1}=0$ is a local maxima and if $\emptyset_{1} \geq \emptyset_{2}$, then $P_{E} \Pi^{\mathrm{p}}\left(\mathrm{x}_{1}\right)$ reaches an interior local maxima at $G^{-1}\left(\emptyset_{1}\right)$; the four cases that are shown in the Figure 1 are explained as :
Case 1: In this case, the optimal solution is obtained by solving the first order conditions, because the value of $P_{E} \Pi^{p}$ [where it is concave and reaches a unique local maxima at $G^{-1}\left(\emptyset_{12}\right)$ ] is less than $G^{-1}\left(\varnothing_{2}\right)$, and therefore eqn. (3) implies that one should stock a positive quantity of both products.
Case 2: In this case, the value of the $G^{-1}\left(\emptyset_{1}\right)$ is greater than $G^{-1}\left(\emptyset_{2}\right)$ [since the only local maxima for this case is $\left.G^{-1}\left(\emptyset_{1}\right)\right]$. Therefore, one should stock only product 1 .
Case 3: $P_{E} \Pi^{p}$ is decreasing in $x_{1}$. Therefore, one should stock only product 2 .
Case 4: In this case, the optimal solution is not completely defined by underage and overage costs, as it depends on the relative values of $P_{E} \Pi^{\mathrm{p}}(0)$ and $P_{E} \Pi^{\mathrm{p}}\left(G^{-1}\left(\emptyset_{1}\right)\right)$ that depend on $G$. [Note that $P_{E} \Pi^{\mathrm{p}}\left(\mathrm{x}_{1}\right)$ has two local maxima at $x_{1}=0$ and $\left.x_{1}=G^{-1}\left(\emptyset_{1}\right)\right]$. Therefore, one should stock both the products.

### 2.2 Dynamic programming formulation

The problem reduces to a two-product problem if the quantities $q_{1}, \ldots \ldots \ldots . . . . ., q_{n-2}$ are fixed in the n -product problem Using backward substitution, we cannot solve the association of a unique fractal of the distribution $F$, with the optimal quantity of $(n-1)$ products, due to the case 4 in figure 1.

The n-product assortment planning and inventory management problem can be formulated as a dynamic program, suggested by the preference ordering in the homogeneous population model. Let $e_{k}\left(x_{k}, X\right)$ be the expected profit from product k when the inventory of product k is $x_{k}$, and the total inventory of products 1 , $\qquad$ $\mathrm{k}-1$ is $X \equiv \sum_{i=1}^{k-1} x_{i}$. We get

$$
\begin{align*}
& e_{k}\left(x_{k}, X\right)=u_{k} x_{k}-\left(u_{k}+o_{k}\right) x_{k} G(x)-\left(u_{k}+o_{k}\right) \int_{X}^{X+x_{k}}\left(X+x_{k}-v\right) g(v) d v \\
& \quad=e_{k}\left(x_{k}+X, 0\right)-e_{k}(X, 0) \tag{6}
\end{align*}
$$

Also, let $e_{k}(x) \equiv e_{k}(x, 0)$ be the expected newsvendor profit when x of product k are stocked in a one-product problem. We have

$$
e_{k}(x)=e_{k} x-\left(u_{k}+o_{k}\right) \int_{0}^{x}(x-v) g(v) d v
$$

Also
$e_{k}^{\prime}(x) \equiv \frac{\partial e_{k}(x)}{\partial x}=u_{k}-\left(u_{k}+o_{k}\right) G(x)$
$e_{k}^{\prime \prime}(x) \equiv \frac{\partial^{2} e_{k}(x)}{\partial x^{2}}=-\left(u_{k}+o_{k}\right) g(x)<0$
$e_{k}(x)$ is a exactingly concave function of x with a unique maximum at $G^{-1}\left(\emptyset_{k}\right)$, where

$$
\emptyset_{k}=\frac{u_{k}}{u_{k}+o_{k}} \text { for } k=1, \ldots \ldots, r
$$

Note in the case, where, for $k, i \epsilon(1, \ldots \ldots ., r) ; k \neq i$,

$$
e_{k}(x)-e_{i}(x)=\left(u_{k}-u_{i}\right) x-\left[\left(u_{k}+o_{k}\right)-\left(u_{i}+o_{i}\right)\right] \int_{0}^{x}(x-v) g(v) d v
$$

and,

$$
\begin{align*}
e_{k}^{\prime}(x)-e_{i}^{\prime}(x) & =\frac{\partial\left[e_{k}(x)-e_{i}(x)\right]}{\partial x} \\
= & \left(u_{k}-u_{i}\right)-\left[\left(u_{k}+o_{k}\right)-\left(u_{i}+o_{i}\right)\right] G(x)  \tag{9}\\
& e_{k}^{\prime \prime}(x)-e_{i}^{\prime \prime}(x)=\frac{\partial^{2}\left[e_{k}(x)-e_{i}(x)\right]}{\partial x^{2}} \\
= & -\left[\left(u_{k}+o_{k}\right)-\left(u_{i}+o_{i}\right)\right] g(x) \tag{10}
\end{align*}
$$

If $\left(u_{k}+o_{k}\right) \geq\left(u_{i}+o_{i}\right)$ then $e_{k}(x)-e_{i}(x)$ is a concave function otherwise it is a convex function. Now if $\left(u_{k}+o_{k}\right) \neq\left(u_{i}+o_{i}\right)$ then let us consider:

$$
\emptyset_{k i}=\frac{\left(u_{k}-u_{i}\right)}{\left(u_{k}+o_{k}\right)-\left(u_{i}+o_{i}\right)}
$$

$e_{k}(x)-e_{i}(x)=\left(u_{k}-u_{i}\right)$ attains a unique stationary point at $G^{-1}\left(\emptyset_{k i}\right)$, if $\emptyset_{k i} \geq 0$. This shows that $e_{k}^{\prime}$ and $e_{i}^{\prime}$ are intersecting each other at most once, due to the nature of this property, we pass on this as the at most onetime crossing property.
On considering eqn. (1) and eqn. (vi), eqn. (i) can be written as:
$P_{E} \Pi^{p}(x)=e_{1}\left(x_{1}, 0\right)+\sum_{k=2}^{r} e_{k}\left(x_{k}, \sum_{i=1}^{k-1} x_{i}\right)=e_{1}\left(x_{1}\right)+\sum_{k=2}^{r}\left[e_{k}\left(\sum_{i=1}^{k} x_{i}\right)-e_{k}\left(\sum_{i=1}^{k-1} x_{i}\right)\right]$
No, in the following ways, dynamic programming is devised to determine the inventory levels. Let $E_{k}(x)$ be the maximum expected profit that can be obtained from product $k, \ldots \ldots ., r$ .,$r$ given that total inventory for products $1, \ldots \ldots, k-1$ is X. We have:

$$
\begin{aligned}
& E_{k}(x)=\max _{x_{k} \geq 0}\left[e_{k}\left(x_{k}, X\right)+E_{k+1}\left(X+x_{k}\right)\right] \\
& =\max _{x_{k} \geq 0}\left[e_{k}\left(x_{k}+X\right)-e_{k}(X)+E_{k+1}\left(X+x_{k}\right)\right] \\
& =\max _{x_{k} \geq 0}\left[e_{k}(x)-e_{k}(X)+E_{k+1}(x)\right] \text { for } k=1, \ldots, r \text { and } E_{r+1}(x)=0, \forall X
\end{aligned}
$$

Assume that,

$$
\begin{equation*}
Z_{k}(x)=e_{k}(x)+E_{k+1}(x) \tag{12}
\end{equation*}
$$

And also introducing

$$
\xi_{k}(X)=\max _{x \geq X} Z_{k}(x)
$$

Such that

$$
\begin{align*}
E_{k}(X) & =\max _{x \geq X} Z_{k}(x)-e_{k}(X)  \tag{13}\\
& =\xi_{k}(X)-e_{k}(X) .
\end{align*}
$$

Lemma 1: $P_{E} \Pi\left(x^{*}\right)=E_{1}(0)$.
Proof: (omitted)
We cannot solved the dynamic programming given by eqn. (13) simply, because in this equation, the function $Z_{k}(x)$ is not well-behaved function. Now considering the eqn.(3), from the two-product problem, we get a convex function of x , i.e.,

$$
E_{2}(x)=e_{2}\left\{\left[G^{-1}\left(\emptyset_{2}\right)-x\right]^{+}\right\}
$$

Which gives

$$
Z_{1}\left(x_{1}\right)=e_{1}\left(x_{1}\right)+E_{2}\left(x_{1}\right)
$$

$=P_{E} \Pi^{p}\left(x_{1}\right)$
[from eqn. (4)]
It is the sum of a concave and a convex function. This can be easily seen that this function has multiple local optima in case 4 of the figure 1, though we can solve this dynamic programming efficiently by proving some properties of $E_{k}(X)$.
Now, from preposition 1, the value function has the following piecewise structure for each $k=1, \ldots, r$.

$$
E_{k}(X)=\left\{\begin{array}{cc}
H_{1}^{k}-e_{j_{1}^{k}}(X) & 0 \leq X \leq h_{1}^{k} \\
H_{2}^{k}-e_{j_{2}^{k}}(X) & 0 \leq X \leq h_{2}^{k} \\
\cdot & \cdot  \tag{14}\\
\cdot & \cdot \\
H_{I^{k}}^{k}-e_{j_{I^{k}}^{k}}(X) & h_{I^{k^{k-1}}}^{k}<X<h_{I^{k}}^{k} \\
0 & \\
& X>h_{I^{k}}^{k}
\end{array}\right.
$$

Where $I^{k}$ is the number of breakpoints in the function, the constants $0<h_{1}^{k}<\cdots<h_{I^{k}}^{k}$ are the values of the breakpoints, the constants $H_{1}^{k}, \ldots \ldots, H_{I^{k}}^{k}$ determine the height of each piece function at $X=0$ and the $j_{i}^{k} \epsilon\{k, \ldots \ldots, r\}$ refer to indices of products.
For the simplification of the notation, the superscript $j$ has to be drop for the variables $I, j, h$ and $H$, where they refer to the parameter $E_{k}$.
Preposition 1: For each $k=1, \ldots, r$
(a). $E_{k}(X)$ has the piecewise structure given by eqn. (14)
(b). $E_{k}(X)$ is continuous in $X$.

## Proof:

(a). using induction method,
for $k=r$ we have,

$$
\begin{align*}
& E_{r}(X)=\max _{x \geq X} e_{r}(x)-e_{r}(X) \\
& =\left\{\begin{array}{rr}
e_{r}\left[G^{-1}\left(\emptyset_{r}\right)\right]-e_{r}(x) & \text { if } X \leq G^{-1}\left(\emptyset_{r}\right) \\
o & \text { otherwise }
\end{array}\right. \tag{15}
\end{align*}
$$

$\because e_{r}$ is continuous, therefore, $E_{r}$ is also continuous in $\left[0, G^{-1}\left(\emptyset_{r}\right)\right]$. Also, $\because E_{r}\left[G^{-1}\left(\emptyset_{r}\right)\right]$ equals zero, therefore, $E_{r}$ is continuous at $G^{-1}\left(\emptyset_{r}\right)$. And, thus the first part of the preposition satisfied for $k=r$. $E_{r}$ is also differentiable everywhere. But we will show that $E_{k}, k<r$, may have points of nondifferentiability.

Figure 3.2: Case 1 and Case


By considering $I^{r}=1, j_{1}^{r}=r, H_{1}^{r}=e_{r}\left[G^{-1}\left(\emptyset_{r}\right)\right]$ and $h_{1}^{r}=G^{-1}\left(\emptyset_{r}\right)$, the second part of the preposition is obtained.
For $E_{k}, k=1, \ldots, r-1$, assuming that it is true for $E_{k+1}$.
Also, $Z_{k}(x)=E_{k+1}(x)+e_{k}(x)$ is continuous in $x . \because E_{k+1}$ is not concave in $x$, therefore, $Z_{k}(x)=$ $E_{k+1}(x)+e_{k}(x)$ may have multiple local optima.
Let us assume the following:

- $\xi_{k}(X)=\max _{x \geq X} Z_{k}(x)$
- $v_{1}$ denote the highest local maxima of $Z_{k}$.
- $v_{2}$ denotes the next highest local maxima of $Z_{k}$ that is located to the right of $v_{1}$ and so on until $v_{s}$ where $s$ is the number of local maxima on the right of $v_{s}$.
- $v_{0}=0$
- For each local maxima $v_{l}$, let $d_{l}$ be such that $v_{l-1}<d_{l}<v_{l}$ and $Z_{k}\left(d_{l}\right)=Z_{k}\left(v_{l}\right)$.
- $\because Z_{k}$ is continuous, there might not exist a point $d_{1}$, however, $d_{2}, \ldots \ldots ., d_{s}$ always exist and are uniquely defined.
- Case 1 is the case when $d_{1}$ exists.
- Case 2 is the case when $d_{1}$ does not exists.

The construction of the values $v_{l}, d_{l}$, in both cases are mentioned in the Figure 3.2.
In case $1, \xi_{k}(X)$ is defined as:

$$
\xi_{k}(X)=\left\{\begin{array}{cl}
E_{k+1}(X)+e_{k}(X), & \text { if } v_{0} \leq X \leq d_{1} \\
E_{k+1}(X)+e_{k}(X), & \text { if } v_{l-1}<X \leq d_{l} ; \text { for } l=2, \ldots \ldots \ldots, s  \tag{16}\\
E_{k+1}\left(v_{l}\right)+e_{k}\left(v_{l}\right), & \text { if } d_{l}<X \leq v_{l} ; \text { for } l=2, \ldots \ldots ., s \\
E_{k+1}(X)+e_{k}(X), & \text { if } v_{s}<X
\end{array}\right.
$$

And in case $2 \xi_{k}(X)$ is defined as:

$$
\xi_{k}(X)=\left\{\begin{array}{cl}
E_{k+1}\left(v_{1}\right)+e_{k}\left(v_{1}\right), & \text { if } v_{0} \leq X \leq v_{1} \\
E_{k+1}(X)+e_{k}(X), & \text { if } v_{l-1}<X \leq d_{l} ; \text { for } l=2, \ldots \ldots ., s \\
E_{k+1}\left(v_{r}\right)+e_{k}\left(v_{r}\right), & \text { if } d_{l}<X \leq v_{l} ; \text { for } l=2, \ldots \ldots, s \\
E_{k+1}(X)+e_{k}(X), & \text { if } v_{s}<X
\end{array}\right.
$$

It shows that in the segments $\left(v_{l-1}, d_{l}\right)$ for $l=1, \ldots \ldots ., s$ and in $\left(v_{s}, \infty\right)$, the value of $\xi_{k}(X)$ is decreasing and is equal to $Z_{k}(X)$, whereas, it is constant in segments $\left(d_{l}, v_{l}\right)$ for $l=1$,., $\qquad$ Therefore, in case 1, the obtained value of $E_{k}$ is:

$$
E_{k}(X)= \begin{cases}E_{k+1}(X) & \text { if } 0 \leq X \leq d_{1} \text { or } v_{l-1}<X \leq d_{l} ; \text { for } l=2, \ldots \ldots \ldots, s \\ E_{k+1}\left(v_{1}\right)+e_{k}\left(v_{1}\right)-e_{k}(X) & \text { if } d_{l}<X \leq v_{l} ; \text { for } l=2, \ldots \ldots \ldots, s \\ E_{k+1}(X) & \text { if } v_{s}<X\end{cases}
$$

Similarly for case 2 , assuming $F_{l}=E_{k+1}\left(v_{l}\right)+e_{k}\left(v_{l}\right) ; l=1$, $\qquad$ $s$ and also $F_{l}$ does not depend on $X$. Therefore, $E_{k}(X)=F_{l}-e_{k}(X)$ for $X \epsilon\left(d_{l}, v_{l}\right] ; l=1, \ldots \ldots . .$, $s$. Using induction hypothesis $E_{k+1}(X)$ is
also decomposed into pieces of the type $H_{i}-e_{j i}(X)$. Considering these together, we obtain a decomposition of $E_{k}$ into pieces of the type $H_{i}-e_{j i}(X)$, where $d_{l}$ and $v_{l}$ are the breakpoints for $l=1, \ldots \ldots ., s$ and the breakpoints of $E_{k+1}$ located in $\left(v_{l-1}, d_{l}\right)$ for $l=1, \ldots \ldots ., s$ and in $\left(v_{s}, \infty\right)$. By using contradiction method we will show that $E_{k}$ is continuous.
Let

$$
\begin{aligned}
& \xi_{k}(X)=\max _{x \geq X} Z_{k}(x) \\
& \equiv \max _{x \geq X}\left[E_{k+1}(x)+e_{k}(x)\right]
\end{aligned}
$$

is not continuous at some $X_{1}, \xi_{k}$ is non-increasing in $X$ because the feasibility set $\{x: x \geq X\}$ gets smaller as $X$ increases. Therefore, at $X_{1}$, we must have (these limits exists because the function is monotone)

$$
\xi_{k}\left(X_{1}^{-}\right)=\xi_{k}\left(X_{1}\right)
$$

and hence,
$\xi_{k}\left(X_{1}^{+}\right) \geq Z_{k}\left(X_{1}^{+}\right)$from the definition of $\xi$, which is a contradiction, $\because Z_{k}$ is continuous. Therefore, $\xi_{k}(X)$ is continuous in $X$ and so is $E_{k}(X)=\xi_{k}(X)-e_{k}(X)$. It is observed that the index of the e function (i.e., $j_{i}$ for some $i$ ) at a certain $X$ can either be replaced with $k$ or remain at its previous value, that is greater than $k$, when going from $E_{k+1}$ to $E_{k}$. On performing the backward induction of the dynamic programming, the index of the product that determine the slope of the value function at a given $X$ can only decrease, therefore, it is referred as the "non-increasing index" property.

### 2.3. Convexity of the value function

We found that the value function in not concave, with the example of a two-product problem, and we will prove that it is actually convex and decreasing in X . For that firstly we show that there are two types of breakpoints in the value function. Let $h_{i}$ is a differentiable breakpoint (DBP) of $E_{k}$ if $E_{k}^{\prime}$ is continuous at $h_{i}$ otherwise it is a non-differentiable breakpoint (NDBP).
The following Lemma shows that for the right derivative at NDBP's is strictly greater to its left derivative. This lemma is useful in solving the dynamic program because it implies that the NDBP's of $E_{k+1}$ cannot constitute local maxima of $Z_{k}$ and therefore interior local maxima can only be found at stationary points of $Z_{k}$.

Lemma 2. For each $h_{i}$ in (xiv),

$$
\begin{equation*}
E_{k}^{\prime}\left(h_{i}^{-}\right) \leq E_{k}^{\prime}\left(h_{i}^{+}\right) \tag{18}
\end{equation*}
$$

Proposition 2. $E_{k}(X)$ is convex and decreasing in X .

### 2.4 Algorithm

It follows from proposition 1, that to each stage of the dynamic programming, we need to consider the following:

1) The vector product indices $\left(j_{1}, \ldots \ldots \ldots . . . . j_{I}\right)$.
2) The vector of breakpoints $\left(h_{1}, \ldots \ldots \ldots . . ., h_{I}\right)$.
3) The first constant $H_{1}$ because by continuity of $E_{k}$.

Now $H_{2}, \ldots \ldots ., H_{I}$ can be derived by using the recursion: $H_{i}=H_{i-1}-e_{j_{i-1}}\left(h_{i-1}\right)+e_{j_{i}}\left(h_{i-1}\right)$ for $i=2, \ldots . I$. Therefore, it is possible to compute $E_{k}(X)$ for every value of $X$ for a given $\left(j_{1}, \ldots \ldots \ldots, j_{I}\right),\left(h_{1}, \ldots \ldots ., h_{I}\right)$ and $H_{1}$. It is found that the NDBP's of $E_{k+1}$ cannot constitute local maxima of $Z_{k}$ (by lemma 1), therefore, we need to consider only stationary points of $Z_{k}$ when we search for local maxima. Due to the fact that $e_{k}$ is a concave function for $k=1, \ldots \ldots . . r$ and of the piecewise structure of $E_{k}$ established in Preposition 1, there exists at most one stationary point between every pair of breakpoints of $E_{k+1}$. Hence to solve the dynamic programming, we propose the following algorithm. Let $j_{I+1}$ be a dummy product with overage product with overage cost and underage cost equal to zero so that $\emptyset_{k, j_{I+1}}=\emptyset_{k}$.
Algorithm 1: For $k=1, \ldots \ldots . . r$, repeat the following steps, given at step $k,\left(j_{1}, \ldots \ldots \ldots . . j_{I}\right)$, $\left(h_{1}, \ldots \ldots . ., h_{I}\right)$ and $H_{1}$ corresponding to $E_{k+1}$.

STEP 1: For $i=1, \ldots \ldots I+1$ such that $u_{k}+o_{k}>u_{j_{i}}+o_{j_{i}}$. Check if $h_{i-1}<G^{-1}\left(\emptyset_{k}, j_{i}\right) \leq h_{i}$. Let $w_{1} \leq \cdots \cdots \leq w_{m}$ be the set of values $G^{-1}\left(\emptyset_{k}, j_{i}\right)$ such that this condition is satisfied.

STEP 2: Let

$$
v_{1}=\operatorname{argmax}_{w_{m}}\left\{Z_{k}\left(w_{m}\right), m=1,,,, M\right\} \text { for } l=2, \ldots \ldots \ldots, s .
$$

Let

$$
v_{r}=\arg \max _{w_{m}}\left\{Z_{k}\left(w_{m}\right) ; w_{m}>v_{l-1}\right\}
$$

such that $v_{s}=w_{m}$.
STEP 3: If $Z_{k}\left(v_{1}\right)<H_{1}$ then this is case 1 of Figure 2. Otherwise it is case 2.
In case 1 , find $d_{1}<v_{1}$ such that $Z_{k}\left(d_{1}\right)=Z_{k}\left(v_{1}\right)$. In both cases, find $v_{l-1}<d_{l}<v_{l}$ and $Z_{k}\left(d_{l}\right)=Z_{k}\left(v_{l}\right)$, for $l=2, \ldots \ldots, s$.

STEP 4: For $l=1, \ldots \ldots . ., s$, find $b_{l}$ such that $h_{b_{l}}<d_{l}<h_{b_{l}+1}$ and $i_{l}$ such that $h_{i_{l}-1}<v_{l}<h_{i_{l}}$.
The new vector of breakpoint is:

$$
\begin{array}{r}
\left(h_{1}, \ldots \ldots, h_{b_{1}}, d_{1}, v_{1}, h_{i_{1}}, \ldots \ldots, h_{b_{2}}, d_{2}, \ldots \ldots, v_{s}, h_{i_{s}}, \ldots \ldots, h_{I}\right) \\
\left(h_{1}, \ldots \ldots ., h_{b_{1}}, d_{1}, v_{1}, h_{i_{1}}, \ldots \ldots, h_{b_{2}}, \ldots \ldots, d_{2}, \ldots \ldots, v_{s}\right) \\
\left(v_{1}, h_{i_{1}}, \ldots \ldots, h_{h_{2}}, d_{2}, \ldots \ldots, v_{s}, h_{i_{s}}, \ldots ., h_{1}\right) \\
\left(v_{1}, h_{i_{1}}, \ldots \ldots, h_{b_{2}}, d_{2}, \ldots \ldots, v_{s}\right)
\end{array}
$$

in case 1, if $v_{s}<h_{I}$ in case 1, if $\quad v_{s} \geq h_{I}$ in case 2 , if $\quad v_{s}<h_{I}$ in case 2 , if $\quad v_{s} \geq h_{I}$

The new vector of product indices is:

$$
\begin{array}{r}
\left(j_{1}, \ldots \ldots, j_{b_{1}}, j_{b_{1}+1}, k, j_{i_{1}}, \ldots \ldots, j_{b_{2}}, j_{b_{2}+1}, \ldots \ldots, k, j_{i_{s}}, \ldots \ldots, j_{I}\right) \\
\left(j_{1}, \ldots \ldots, j_{b_{1}}, j_{b_{1}+1}, k, j_{i_{1}}, \ldots \ldots, j_{b_{2}}, j_{b_{2}+1}, \ldots \ldots, k\right) \\
\left(k, j_{i_{1}}, \ldots \ldots, j_{b_{2}}, j_{b_{2}+1}, \ldots \ldots, k, j_{i_{s}}, \ldots \ldots, j_{I_{2}}\right) \\
\left(k, j_{i_{1}}, \ldots \ldots, j_{b_{2}}, j_{b_{2}+1}, \ldots \ldots, k\right)
\end{array}
$$

$$
\text { in case } 1, \text { if } \quad v_{s}<h_{I}
$$

$$
\text { in case } 1, \text { if } v_{s} \geq h_{I}
$$

$$
\text { in case } 2 \text {, if } v_{s}<h_{I}
$$

$$
\text { in case } 2 \text {, if } \quad v_{s} \geq h_{I}
$$

In case 1 , the value of $H_{1}$ remains unchanged. In case 2 , the value of $H_{1}$ is replaced by $Z_{k}\left(v_{1}\right)$.
Proof: From proposition 1, step 2 to step 4 are followed directly and in these steps it is described that it is enough to look at $G^{-1}\left(\emptyset_{k}, j_{i}\right) \quad$ for $i=1, \ldots \ldots, I+1$ when searching for the local maxima $v_{1}, \ldots \ldots, v_{s}$ of $Z_{k}$ as defined in eqn. (16). If $v$ is a local maxima of $Z_{k}^{\prime}(v)=0$ [from the proof of lemma 3.4]. By proposition 1 , we have

$$
Z_{k}(x)=\left\{\begin{array}{cc}
H_{1}-e_{j_{1}}(x)+e_{k}(x) & 0 \leq x \leq h_{1} \\
H_{2}-e_{j_{2}}(x)+e_{k}(x) & h_{1} \leq x \leq h_{2} \\
\cdot & \cdot \\
\cdot & \cdot \\
\cdot & \cdot \\
H_{\mathrm{I}}-e_{j_{I}}(x)+e_{k}(x) & h_{I-1}<x \leq h_{I} \\
e_{k}(x) & x>h_{I}
\end{array}\right.
$$

At any $x$, where $Z_{k}$ is differentiable, $Z_{k}^{\prime}(x)=e_{k}^{\prime}(x)-e_{j_{i}}^{\prime}(x)$ for some $i$.
And $e_{k}^{\prime}(x)-e_{j_{i}}^{\prime}(x)=0$ gives $x=G^{-1}\left(\emptyset_{k, j_{i}}\right)$.
As the result, this point is a local maxima if $e_{k}^{\prime \prime}(x)-e_{j_{i}}^{\prime \prime}(x) \geq 0$, that is, if $u_{k}+o_{k}>u_{j_{i}}+o_{j_{i}}$.
With the help of this algorithm the value of the optimal expected profit can be obtained.
Then, the optimal stocking quantities $x_{k}^{*}$ for $k=1, \ldots \ldots, r$ can be recovered from $\left(j_{1}^{k}, \ldots \ldots, j_{I^{k}}^{k}\right),\left(h_{1}^{k}, \ldots \ldots, h_{I^{k}}^{k}\right)$ for $k=1, \ldots \ldots, r$ in the following way:

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## Algorithm 2:

* Set $X=0, k=1$.
* While $X \leq h_{I^{k}}^{k}$,
- Find $i$ such that $h_{i-1}^{k}<X \leq h_{i}^{k}$.
- If $j_{i}^{k}=k$ then $x_{k}^{*}=h_{i}^{k}-X$ and $X \rightarrow X+x_{k}^{*}$.

$$
\text { Otherwise, } x_{k}^{*}=0 .
$$

- If $k \leq r, x_{k}^{*}=\cdots \cdots=x_{r}^{*}=0$.

Proof: From using eqn.(13) we get that the optimal quantity of product $k$, as function of $X=x_{1}^{*}+\cdots \cdots+x_{k-1}^{*}$ is given by

$$
\begin{equation*}
x_{k}^{*}=\operatorname{argmax}_{x \geq X} Z_{k}(x)-X \tag{19}
\end{equation*}
$$

Let $j_{i}^{k}$ be such that $E_{k}(X)=H_{j_{i}^{k}}-e_{j_{i}^{k}}(X)$ for $i \in\left\{1, \ldots \ldots, I^{k}\right\}$. From (16), we know that $\max _{x \geq X} Z_{k}(x)$ can be either equal to $Z_{k}(x)$ or $Z_{k}\left(v_{l}\right)$ where $v_{l}>X$ a local maximum of is $Z_{k}$. In the first case we have $E_{k}(X)=E_{k+1}(X)$, so that $j_{i}^{k} \neq k$ and eqn. (19) gives that $x_{k}^{*}=0$. In the second case, we have $E_{k}(X)=Z_{k}\left(v_{l}\right)-e_{k}(X)$, so that $j_{i}^{k}=k$ and eqn. (19) gives that $x_{k}^{*}=v_{l}-X$. At last by proposition 1, we have that $h_{i}^{k}=v_{l}$.

Lemma 3: Let $K=\left\{k_{1}, \ldots \ldots, k_{s}\right\} \subseteq\{1, \ldots \ldots, r\}$ be the set such that $x_{k}^{*}>0$ if $k \in K$ and $x_{k}^{*}=0$, otherwise assume that $k_{1}<k_{2}<\cdots \cdots<k_{s}$. Then

$$
x_{k_{1}}^{*}=\left\{\begin{array}{cl}
G^{-1}\left(\emptyset_{k_{m}, k_{m}+1}\right)-\sum_{i=1}^{m-1} x_{k_{i}}^{*} & m=1, \ldots \ldots, s-1 \\
G^{-1}\left(\emptyset_{k_{s}}\right)-\sum_{i=1}^{s-1} x_{k_{i}}^{*} & m=s
\end{array}\right.
$$

Also, for $=1, \ldots \ldots, s-1$,

$$
\begin{aligned}
& u_{k_{m}}+o_{k_{m}}>u_{k_{m}+1}+o_{k_{m}+1} \\
& u_{k_{m}}>u_{k_{m}+1} \\
& \emptyset_{k_{m}}<\emptyset_{k_{m}+1}
\end{aligned}
$$

Proof: (omitted)
Let $\left(\emptyset_{k_{1}, k_{2}}, \ldots \ldots, \emptyset_{k_{s}-1, s}, \emptyset_{k_{s}}\right)$ is the set of optimal critical fractiles corresponding to optimal solution $x^{*}$. We know that $\xi(v)$ denotes the marginal expected profit of the $v$-th potential unit of demand (from 3.1). From that $\xi(v)$ is a continuous decreasing and piecewise convex function given by:

$$
Z_{k}(x)=\left\{\begin{array}{cc}
u_{k_{1}}-\left(u_{k_{1}}+o_{k_{1}}\right) G(v) & v \leq x_{k_{1}} \\
u_{k_{2}}-\left(u_{k_{2}}+o_{k_{2}}\right) G(v) & x_{k_{1}}<v \leq x_{k_{1}}+x_{k_{2}} \\
\cdot & \cdot \\
\cdot & \cdot \\
\cdot & \cdot \\
\cdot & \sum_{k=1}^{s-1} x_{k}<v \leq \sum_{k=1}^{s} x_{k} \\
u_{k_{s}}-\left(u_{k_{s}}+o_{k_{s}}\right) G(v) & v>\sum_{k=1}^{s} x_{k}
\end{array}\right.
$$

The most preferred product of all customers has the highest values of risk and return, their second choice has the second highest values of risk and return etc i.e., in the optimal assortment, the preference order matches the risk and return order(lemma 3 also refer this).

### 2.5 Complexity of the Algorithm

To obtain the complexity of Algorithm 1 we have to establish a bound on $I^{k}$ which is the number of breakpoints in the value function.

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Lemma 4. $I^{k} \leq 2(r-k)+1$, where $I^{k}$ is defined in eqn. (14). The indices $j_{1}^{k}, \ldots \ldots, j_{I^{k}}^{k}$ of the e function in each piece of the value function in eqn.(14) belong to the set $\{k, \ldots \ldots, r\}$. However, two indices can correspond to the same product that is we can have $j_{i}^{k}=j_{y}^{k}$ for $1 \leq i<y \leq I^{k}$. In order to obtain a bound on $I^{k}$, we established that between two repetitions of the same product in the sequence of indices, there should be at least one product that did not anywhere before in the sequence. This creates a limit on the number of repetitions of one product since there could be at most $r-k+1$ different products in the sequence. In the following we assume that a line search is assumed to be O(1).
Proposition 3. The complexity of the algorithm is $O\left(n^{2}\right)$.
Proof. In the Algorithm 1, Step 1 has to be repeated at most $I^{k}$ number of times. For Step 2:

* Set $s=1, v_{s}=w_{m}$ and max $=Z_{k}\left(v_{s}\right)$
* For $m=M \rightarrow 1$
- If $Z_{k}\left(w_{m}\right)>\max$ then $s=s+1, v_{s}=w_{m}$ and $\max =Z_{k}\left(v_{s}\right)$

At the end, in step 3, we come to know that to search $d_{l}$ takes $O(1)$ time. Since all the steps are repeated n times therefore all steps can be done in $O(n)$ times which results that the complexity is $O\left(n^{2}\right)$.

### 2.6 Static versus Dynamic substitution

Under this section, we compare the expected profit obtained under the assumption of static substitution with that obtained under the assumption of dynamic substitution in the homogeneous population setting. Mainly, by considering dynamic substitution, we measure the percentage increase in expected profit:

$$
f=\frac{P_{E} \Pi^{P}\left(x^{* P}\right)-P_{E} \Pi^{S}\left(x^{* S}\right)}{P_{E} \Pi^{S}\left(x^{* S}\right)}
$$

Where $\Pi^{S}$ and $x^{* S}$ denotes the profit and optimal inventory vector under static substitution respectively. Therefore, all customers have the same preferences and do not substitute in the store, the optimal assortment under static substitution, contain only one product, which is the one with the largest expected profit. Consider the assumption of dynamic substitution $k^{*}$ be the lowest product index such that

$$
\begin{gathered}
k=\arg \max _{k} e_{k}\left\{G^{-1}\left(\emptyset_{k}\right)\right\} \\
x_{k}^{* S}= \begin{cases}G^{-1}\left(\emptyset_{k}\right) & \text { if } k=k^{*} \\
0 & \text { otherwise }\end{cases}
\end{gathered}
$$

Lemma 5. When $\mathrm{r}=2$, f can be made arbitrarily close to 1 .
Proof. Consider $\mathrm{r}=2$ and also an example where $f \rightarrow 1$. Fix $u_{1}, o_{1}, u_{2}$ such that $u_{1}>u_{2}$ then let $o_{2}$ such that:

$$
\begin{equation*}
e_{1}\left(G^{-1}\left(\emptyset_{1}\right)\right)=e_{2}\left(G^{-1}\left(\emptyset_{2}\right)\right) \tag{20}
\end{equation*}
$$

This implies that $\emptyset_{1}<\emptyset_{2}$ and the underage and overage costs fall into Case 1 of Figure 1.
Case 1 being the case where it is optimal to stock both products, we get

$$
f=\frac{e_{1}\left(G^{-1}\left(\emptyset_{12}\right)\right)-e_{2}\left(G^{-1}\left(\emptyset_{12}\right)\right)}{e_{1}\left(G^{-1}\left(\emptyset_{1}\right)\right)}
$$

Now if we let $o_{1} \rightarrow \infty$ and $u_{2}, o_{2} \rightarrow 0$ such that $\emptyset_{1} \rightarrow 0$ and $\emptyset_{2} \rightarrow 1$ while making sure that condition (20) holds. This implies that:

$$
\begin{aligned}
& e_{1}\left(G^{-1}\left(\emptyset_{12}\right)\right)-e_{2}\left(G^{-1}\left(\emptyset_{12}\right)\right) \\
& =\left(u_{1}-u_{2}\right) G^{-1}\left(\emptyset_{12}\right)-\left[\left(u_{1}+o_{1}\right)-\left(u_{2}+o_{2}\right)\right] \int_{0}^{G^{-1}\left(\emptyset_{12}\right)}(x-v) g(v) d v \\
& \rightarrow u_{1} G^{-1}\left(\emptyset_{1}\right)-\left(u_{1}+o_{1}\right) \int_{0}^{G^{-1}\left(\emptyset_{1}\right)}(x-v) g(v) d v \\
& =e_{1}\left(G^{-1}\left(\emptyset_{1}\right)\right) \quad \text { so that } f \rightarrow 1
\end{aligned}
$$

With $r$ products the percentage increase $f$ can be larger than $100 \%$. This shows that assuming static substitution can cause a substantial drop in expected profit because the retailer is not able to take advantage of the differences in return and risk between products.

## THE NESTED PREFERENCES MODEL

In this, customers can be of the following types: (1), ( 1,2 ), or $(1,2, \ldots \ldots, r)$.

### 3.1 Trend-following partitioning

Let us consider that with probability $\delta_{1}$ all customers are of type (1), with probability $\delta_{2}$ they are of type $(1,2), \ldots \ldots$. so on and with probability $\delta_{r}$ they are of type ( $1,2, \ldots \ldots, r$ ). Let $\sum_{k=1}^{r} \delta_{k}=1$ and $\gamma_{k}=\sum_{i=k}^{r} \delta_{i}$ be the probability that a customer is willing to buy product $k$. This gives $\gamma_{1}=1$ and $1 \geq$ $\gamma_{2} \geq \cdots \cdots \geq \gamma_{r}$. The demand for product $k$ is denoted by $D_{k}^{N T}$, where NT stands for Nested preferences with trend-following partitioning and is given by:

$$
D_{k}^{N T}=\left\{\begin{array}{c}
{\left[D-\sum_{j=1}^{k-1} x_{j}\right.} \\
0
\end{array}\right.
$$

$$
D_{1}^{N T}=D
$$

$$
\text { with probability } \gamma_{k} ; k=2, \ldots \ldots ., r
$$

$$
\text { with probability }\left(1-\gamma_{k}\right) ; k=2, \ldots \ldots, r
$$

Let $\prod^{N T}(x, u, o)$ denotes the profit in this setting under the continuous approximation of demand, for inventory vector $x$ and underage and overage cost vectors $u$ and $o$ respectively.

Lemma 6. $P_{E} \Pi^{N T}(x, u, o)=P_{E} \Pi^{P}(x, \bar{u}, o)$ where $\overline{u_{k}}=\gamma_{k} u_{k}-\left(1-\gamma_{k}\right) o_{k}$ for $k=1, \ldots \ldots, r$.

$$
\begin{aligned}
& \text { Proof. } P_{E} \Pi^{N T}(x, u, o)=\sum_{k=1}^{r}\left\{u_{k} x_{k}-\left(u_{k}+o_{k}\right) P_{E}\left[x_{k}-D_{k}\right]^{+}\right\} \\
& =\sum_{k=1}^{r}\left\{u_{k} x_{k}-\left(u_{k}+o_{k}\right) x_{k}\left[\left(1-\gamma_{k}\right)+\gamma_{k} G\left(\sum_{j=1}^{k-1} x_{j}\right)\right]-\left(u_{k}+o_{k}\right) \gamma_{k} \int_{\sum_{j=1}^{k-1} x_{j}}^{\Sigma_{j=1}^{k} x_{j}}\left(\sum_{j=1}^{k} x_{j}-v\right) g(v) d v\right\} \\
& =\sum_{k=1}^{r}\left[\left(\gamma_{k} u_{k}-\left(1-\gamma_{k}\right) o_{k}\right) x_{k}-\gamma_{k}\left(u_{k}+o_{k}\right)\left(x_{k} G\left(\sum_{j=1}^{k-1} x_{j}\right)+\int_{\sum_{j=1}^{k-1} x_{j}}^{\sum_{j=1}^{k} x_{j}}\left(\sum_{j=1}^{k} x_{j}-v\right) g(v) d v\right)\right]
\end{aligned}
$$

On considering $\bar{u}_{k}=\gamma_{k} u_{k}-\left(1-\gamma_{k}\right) o_{k}$, we have

$$
P_{E} \Pi^{N T}(x, u, o)=\sum_{k=1}^{r}\left[\overline{u_{k}} x_{k}-\left(\overline{u_{k}}+o_{k}\right)\left(x_{k} G\left(\sum_{j=1}^{k-1} x_{j}\right)+\int_{\sum_{j=1}^{k-1} x_{j}}^{\sum_{j=1}^{k} x_{j}}\left(\sum_{j=1}^{k} x_{j}-v\right) g(v) d v\right)\right]
$$

$=P_{E} \Pi^{P}(x, \bar{u}, o)$.

### 3.2 Fixed partitioning

Let us consider that a fixed proportion $\delta_{1}$ of customers are of type (1), $\delta_{2}$ are of type ( 1,2 ) , ......, and $\delta_{r}$ are of type $(1,2, \ldots \ldots, r)$. Again we assume that $\sum_{k=1}^{r} \delta_{k}=1$ and $\gamma_{k}=\sum_{i=k}^{r} \delta_{i}$ be the probability that a customer is willing to buy product. Also let $\alpha_{k}=\frac{\delta_{k}}{\delta_{k-1}}$ for $k=2, \ldots \ldots, r$ be the proportion of customers who are willing to buy product $k-1$, who are also willing to buy product $k$. The preferences can be fully characterized by either one of the three vectors: $\left(\delta_{1}, \ldots \ldots, \delta_{r}\right),\left(\gamma_{1}, \ldots \ldots, \gamma_{r}\right)$ or $\left(\alpha_{1}, \ldots \ldots, \alpha_{r}\right)$. The demand for product $k$ is denoted by $D_{k}^{N F}$, where NF stands for Nested preferences with Fixed partitioning, and is given by:

$$
\begin{aligned}
& D_{1}^{N F}=D \\
& D_{k}^{N F}=\left[D_{k-1}-x_{k-1}\right]^{+} \alpha_{k} \\
& =\left[D-\sum_{j=1}^{k-1} \frac{x_{j}}{\gamma_{j}}\right] \gamma_{k} \quad k=2, \ldots \ldots, r
\end{aligned}
$$

Let $\prod^{N F}(x, u, o)$ denote profit in this setting under the continuous approximation of demand, for inventory vector $x$ and underage and overage cost vectors $u$ and $o$ respectively.

Lemma 7. $P_{E} \Pi^{N F}(x, u, o)=P_{E} \Pi^{P}(\bar{x}, \bar{u}, \bar{o})$ where, for $k=1, \ldots \ldots ., r$
$\bar{x}_{k}=\frac{x_{k}}{\gamma_{k}}$

$$
\begin{align*}
& \bar{u}_{k}=u_{k} \gamma_{k}  \tag{21}\\
& \quad \bar{o}_{k}=o_{k} \gamma_{k}
\end{align*}
$$

Proof. $P_{E} \Pi^{N F}(x, u, o)=\sum_{k=1}^{r}\left[u_{k} x_{k}-\left(u_{k}+o_{k}\right) P_{E}\left[x_{k}-D_{k}\right]^{+}\right]$

$$
=\sum_{k=1}^{r}\left[u_{k} x_{k}-\left(u_{k}+o_{k}\right) \int_{0}^{\infty}\left[x_{k}-\left(v-\sum_{j=1}^{k-1} \frac{x_{j}}{\gamma_{j}}\right)^{+} \gamma_{k}\right]^{+} g(v) d v\right]
$$

$$
=\sum_{k=1}^{r}\left[u_{k} x_{k}-\left(u_{k}+o_{k}\right) x_{k} G\left(\sum_{j=1}^{k-1} \frac{x_{j}}{\gamma_{j}}\right)-\left(u_{k}+o_{k}\right) \int_{\sum_{j=1}^{k-1}}^{\infty}\left[x_{k}-\left(v-\sum_{j=1}^{k-1} \frac{x_{j}}{\gamma_{j}}\right) \gamma_{k}\right]^{+} g(v) d v\right]
$$

$$
=\sum_{k=1}^{r}\left[u_{k} x_{k}-\left(u_{k}+o_{k}\right) x_{k} G\left(\sum_{j=1}^{k-1} \frac{x_{j}}{\gamma_{j}}\right)-\left(u_{k}+o_{k}\right) \int_{\sum_{j=1}^{k-1} \frac{x_{j}}{\gamma_{j}}}^{\sum_{j=1}^{k}}\left[x_{k}-\left(v-\sum_{j=1}^{k-1} \frac{x_{j}}{\gamma_{j}}\right) \gamma_{k}\right] g(v) d v\right]
$$

On doing the transformation of variables in eqn.(21), we get:

$$
\begin{gathered}
P_{E} \prod^{N F}(x, u, o)=\sum_{k=1}^{r}\left[\bar{u}_{k} \bar{x}_{k}-\left(\bar{u}_{k}+\bar{o}_{k}\right)\left(\bar{x}_{k} G\left(\sum_{j=1}^{k-1} \bar{x}_{j}\right)+\int_{\sum_{j=1}^{k-1} \frac{x_{j}}{\gamma_{j}}}^{\sum_{j=1}^{k}}\left(\sum_{j=1}^{\frac{x_{j}}{\gamma_{j}}} \bar{x}_{j}-v\right) g(v) d v\right)\right] \\
=P_{E} \prod^{P}(\bar{x}, \bar{u}, \bar{o})
\end{gathered}
$$

### 3.3 Random partitioning

The number of customers of each type, in this case, is a random variable. Let us consider that $R_{k}$ be the random proportion of customers who are willing to buy product $k-1$, who are also willing to buy product $k$ and let $\alpha_{k}=P_{E}\left[R_{k}\right]$ for $k=2, \ldots \ldots, r$. The demand for product $k$ is denoted by $D_{k}^{N R}$, where NR stands for Nested preferences with Random partitioning.

$$
\begin{gathered}
D_{1}^{N R}=D \\
D_{k}^{N R}=\left[D_{k-1}^{N R}-x_{k-1}\right]^{+} R_{k} \\
=\left[D-\sum_{j=1}^{k-1} \frac{x_{j}}{\prod_{i=2}^{j} R_{i}}\right]^{+} \prod_{j=2}^{k} R_{j} \quad k=2, \ldots \ldots, r
\end{gathered}
$$

Lemma 8. $D_{k}^{N F} \leq D_{k}^{N F}$ for $k=1, \ldots \ldots, r$
Proof. The proof can be done by using an induction on $k$ i.e., for $k=1$, we have

$$
D_{1}^{N F}=D_{1}^{N R}=D
$$

Let $D_{k-1}^{N F} \leq D_{k-1}^{N R}$, from the definition of the increasing convex order, it is equivalent to:

$$
P_{E}\left[D_{k-1}^{N R}-x_{k-1}\right]^{+} \geq P_{E}\left[D_{k-1}^{N F}-x_{k-1}\right]^{+} \quad \forall x_{k-1}
$$

then we get:

$$
\begin{gathered}
P_{E}\left[D_{k}^{N R}-x_{k}\right]^{+} \geq P_{E}\left[\left(D_{k-1}^{N R}-x_{k-1}\right)^{+} \alpha_{k}-x_{k}\right]^{+} \\
\geq P_{E}\left[\left(D_{k-1}^{N F}-x_{k-1}\right)^{+} \alpha_{k}-x_{k}\right]^{+} \\
=P_{E}\left[D_{k}^{N F}-x_{k}\right]^{+}
\end{gathered}
$$

The first inequality is an application of Jensen's inequality while the second inequality uses Theorem 3.4.9 of Shaked and Shanthikumar (111994) and the fact that $\left[\left(v-x_{k-1}\right)^{+} \delta_{k}-x_{k}\right]^{+}$is an
increasing convex function of $v$. Which gives that $P_{E}\left[D_{k}^{N F}\right] \leq P_{E}\left[D_{k}^{N R}\right]$ for $k=1, \ldots \ldots$. r. However, for $k=2$, we have the following stronger result.

Lemma 9. If $r=2$
$P_{E}\left[\Pi^{N F}(x)\right] \geq P_{E}\left[\Pi^{N R}(x)\right]$ for every vector $x$.
Proof. We have

$$
\begin{equation*}
P_{E}\left[D_{2}^{N R}\right]=P_{E}\left[\left(D-x_{1}\right)^{+} R_{2}\right]=P_{E}\left[D-x_{1}\right]^{+} \delta_{2}=P_{E}\left[D_{2}^{N F}\right] \tag{22}
\end{equation*}
$$

so that

$$
D_{2}^{N F} \leq D_{2}^{N R}
$$

Which gives that $P_{E}\left[x_{2}-D_{2}^{N F}\right]^{+} \leq P_{E}\left[x_{2}-D_{2}^{N R}\right]^{+}$
and therefore, $\quad P_{E}\left[\Pi^{N F}(x)\right] \geq P_{E}\left[\Pi^{N R}(x)\right]$ for every vector $x$.
The products that are stocked in a positive quantity in $x$, we obtain for $r>2$, the same result given a certain condition on the price and cost parameters
Proposition 4. For a given vector $x$, let $K=\left\{k_{1}, \ldots \ldots, k_{s}\right\} \subseteq\{1, \ldots \ldots, r\}$ be such that $x_{k}>0$ if $k \in K$ and $x_{k}=0$ otherwise. Assume that $k_{1}<k_{2}<\cdots \cdots<k_{s}$. If form $=1, \ldots \ldots, s-1$
then

$$
\begin{align*}
& u_{k_{m}}+o_{k_{m}}>\frac{\gamma_{k_{m+1}}}{\gamma_{k_{m}}}\left(u_{k_{m+1}}+o_{k_{m+1}}\right)  \tag{23}\\
& P_{E}\left[\Pi^{N F}(x)\right] \geq P_{E}\left[\Pi^{N R}(x)\right]
\end{align*}
$$

Proof. We have:

$$
\begin{aligned}
& P_{E} \Pi^{N R}(x)=\sum_{k=1}^{r}\left[u_{k} x_{k}-\left(u_{k}+o_{k}\right) P_{E}\left[x_{k}-D_{k}^{N R}\right]^{+}\right] \\
& =\sum_{m=1}^{s}\left[u_{k_{m}} x_{k_{m}}-\left(u_{k_{m}}+o_{k_{m}}\right) P_{E}\left[x_{k_{m}}-D_{k_{m}}^{N R}\right]^{+}\right] \\
& =\sum_{m=1}^{s}\left[\left(u_{k_{m}}+o_{k_{m}}\right)\left(P_{E}\left[D_{k_{m}}^{N R}\right]-P_{E}\left[D_{k_{m}}^{N R}-x_{k_{m}}\right]^{+}\right)-o_{k_{m}} x_{k_{m}}\right] \\
& =\left(u_{k_{m}}+o_{k_{m}}\right)\left(P_{E}[D]-P_{E}\left[D-x_{k_{m}}\right]^{+}\right) \\
& +\sum_{m=2}^{s}\left[\left(u_{k_{m}}+o_{k_{m}}\right)\left(P_{E}\left[D_{k_{m-1}}^{N R}-x_{k_{m-1}}\right]+\frac{\gamma_{k_{m}}}{\gamma_{k_{m-1}}}-P_{E}\left[D_{k_{m}}^{N R}-x_{k_{m}}\right]^{+}\right)\right]-\sum_{m=1}^{s} o_{k_{m}} x_{k_{m}} . \\
& =\left(u_{k_{m}}+o_{k_{m}}\right) P_{E}[D]-\sum_{k=2}^{r}\left[\left(u_{k_{m-1}}+o_{k_{m-1}}\right)-\frac{\gamma_{k_{m}}}{\gamma_{k_{m-1}}}\left(u_{k_{m}}+o_{k_{m}}\right)\right] P_{E}\left[D_{k_{m-1}}^{N R}-x_{k_{m-1}}\right]^{+} \\
& \quad-\left(u_{k_{s}}+o_{k_{s}}\right) P_{E}\left[D_{k_{s}}^{N R}-x_{k_{s}}\right]^{+}-\sum_{m=1}^{s} o_{k_{m}} x_{k_{m}}
\end{aligned}
$$

$$
\text { Also, } P_{E}\left[D_{k}^{N R}-x_{k}\right]^{+} \geq P_{E}\left[D_{k}^{N F}-x_{k}\right]^{+} \forall k \& x \text {, we get the desired result, if }
$$

$$
\left(u_{k_{m-1}}+o_{k_{m-1}}\right)-\frac{\gamma_{k_{m}}}{\gamma_{k_{m-1}}}\left(u_{k_{m}}+o_{k_{m}}\right) \geq 0 \text { for } m=1, \ldots \ldots, s-1
$$

Proposition 5. If $x^{* N R}$ satisfies equation (23), then $P_{E} \prod^{N F}\left(x^{* N F}\right) \geq P_{E} \Pi^{N R}\left(x^{* N R}\right)$.
Proof. (omitted)
We propose using $x^{* N R}$ as a heuristic for the NR setting. We can estimate the performance of that solution, that is $P_{E} \Pi^{N R}\left(x^{* N R}\right)$ by comparing it to the upper bound $P_{E} \prod^{N F}\left(x^{* N F}\right)$. Numerical results are presented in section 6 .

## OUTTREE-SHAPED PREFERENCES

The preference structure can be represented by an out tree where the products are shown by the nodes. From the word 'outtree' it is clear that there is a single initial note representing the first choice product for all consumer types and there is a unique directed path from the initial node to any other node. Each such path corresponds to a consumer type. The following figure shows an example of such a tree:


Figure 4: Example of Outtree
In the above Figure, the initial node (the one with no predecessor), correspond to product 1. For node , let $S_{k}$ be the sets of direct successors of node $k$. By definition of an outtree, nodes have only one immediate predecessor, let $p(k)$ be the predecessor of node. Finally let $P_{k}$ be the set of (nonimmediate) predecessors of node $k$.

### 4.1 Fixed partitioning

Let the fixed proportion of customers wanting to buy product $p(k)$ who are also willing to buy product $k$ be defined by $\alpha_{k}$ (for $k>1$ ). Then we get:
$\sum_{i \in S_{k}} \alpha_{i} \leq 1, k=2, \ldots \ldots, r$ and let $\alpha_{1}=1, \alpha_{2}+\alpha_{3} \leq 1, \alpha_{4}+\alpha_{5} \leq 1 \& \alpha_{6} \leq 1$. Also let us consider $\gamma_{k}=\prod_{j \in P_{k} \cup\{k\}} \alpha_{j}$ as the total proportion of customers that are willing to buy product. $D_{1}^{O F}$ denoted the demand for product $k$ and OF stands for Outtree-shaped preferences with Fixed partitioning, and is given by:

$$
\begin{gathered}
D_{1}^{O F}=D \\
D_{k}^{O F}=\left[D_{p(k)}^{O F}-x_{p(k)}\right]^{+} \alpha_{k} \\
=\left[D-\sum_{j \in P_{k}} \frac{x_{j}}{\gamma_{j}}\right]^{+} \gamma_{k} \quad k=2, \ldots \ldots, r
\end{gathered}
$$

Left with the transformation that is similar to that of section 4.2,

$$
\begin{gathered}
\bar{x}_{k}=\frac{x_{k}}{\gamma_{k}} \\
\bar{u}_{k}=u_{k} \gamma_{k} \\
\bar{o}_{k}=o_{k} \gamma_{k}
\end{gathered}
$$

Let $\Pi^{O F}$ denote the profit, and then expected profit is given by:

$$
P_{E} \Pi^{O F}(x)=\sum_{k=1}^{r}\left[\bar{u}_{k} \bar{x}_{k}-\left(\bar{u}_{k}+\bar{o}_{k}\right)\left(\bar{x}_{k} G\left(\sum_{j \in P_{k}} \bar{x}_{j}\right)+\int_{\sum_{j \in P_{k}} \bar{x}_{j}}^{\sum_{j \in P_{k}} \bar{x}_{j}+\bar{x}_{k}}\left(\sum_{j \in P_{k}} \bar{x}_{j}+\bar{x}_{k}-v\right) g(v) d v\right)\right]
$$

Now to solve this problem, let us consider the following dynamic programming formulation. Let $E_{k}(x)$ be the maximum expected profit obtained from products in $S_{k} \cup\{k\}$ given that total inventory for products in $P_{k}$ is $X$. We get:

$$
\begin{gathered}
E_{k}(X)=\max _{x_{k} \geq 0}\left\{e_{k}\left(x_{k}, X\right)+\sum_{i \in S_{k}} E_{i}\left(X+x_{k}\right)\right\} \\
=\max _{x \geq X} Z_{k}(x)-e_{k}(X)
\end{gathered}
$$

Where,

$$
Z_{k}(x)=e_{k}(x)+\sum_{i \in S_{k}}^{x \geq x} E_{i}(x)
$$

and we have $P_{E} \Pi^{O F}(x)=E_{1}(0)$. Also in this case, we show that $E_{k}$ has a piecewise structure given by equation (14). The only difference is that the indices $j_{i}$ do not necessarily belong to the set $\{k, \ldots \ldots, r\}$ but rather to a large set let it be $C P_{k}$ which is defined as the set of composite products also let " $k_{1}+\cdots \cdots+k_{s}$ " defined as composite product with underage cost $u_{k_{1}}+\cdots \cdots+u_{k_{s}}$ and overage cost $o_{k_{1}}+\cdots \cdots+o_{k_{s}}$ where, $k_{1} \neq k_{2} \neq \cdots \cdots \neq k_{s}$ and $k_{j} \in\{1, \ldots \ldots, r\}$ for $j=1, \ldots \ldots$, $s$. We have: $e_{"_{k_{1}}+\cdots \cdots+k_{s} "}(x)=\left(u_{k_{1}}+\cdots \cdots+u_{k_{s}}\right) x-\left(u_{k_{1}}+\cdots \cdots+u_{k_{s}}+o_{k_{1}}+\cdots \cdots+o_{k_{s}}\right) \int_{0}^{x}(x-v) g(v) d v$

$$
=e_{k_{1}}(x)+\cdots \cdots+e_{k_{s}}(x)
$$

By eqn. (7) and (8), this is a concave function having maxima at $\emptyset_{k_{1}+\cdots \cdots+k_{m} "}=\frac{u_{k_{1}}+\cdots \cdots+u_{k_{s}}}{u_{k_{1}}+\cdots \cdots+u_{k_{s}}+o_{k_{1}}+\cdots \cdots+o_{k_{s}}}$
And we have:
$C P_{k}=\left\{\mathrm{k}_{1}+\cdots \cdot+\mathrm{k}_{\mathrm{s}}: k_{j} \epsilon\{k, \ldots \ldots, r\}\right.$ for $j=1, \ldots \ldots .$, , there is no any path connecting any two $\left.k_{j}\right\}$
Proposition 6. For each $k=1, \ldots \ldots, r$,
(1) $E_{k}(X)$ has the piecewise structure given by (14)
(2) $E_{k}(X)$ is continuous in $X$

Proof. The proof of continuity is similar to that of Proposition 1. Following the same arguments as in Proposition 1, we obtain (in Case 1):

$$
E_{k}(x)= \begin{cases}\sum_{m \in S_{k}} E_{m}(X) & \text { if } 0 \leq X \leq d_{1} \\ \sum_{m \in S_{k}} E_{m}\left(v_{1}\right)+e_{k}\left(v_{1}\right)-e_{k}(X) & \text { if } d_{1} \leq X \leq v_{1} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \sum_{m \in S_{k}} E_{m}\left(v_{s}\right)+e_{k}\left(v_{s}\right)-e_{k}(X) & \text { if } d_{s} \leq X \leq v_{s} \\ \sum_{m \in S_{k}} E_{m}(X) & \text { if } v_{s}<X\end{cases}
$$

For $l=1, \ldots \ldots ., s ; C P_{l}=\sum_{i \in S_{k}} E_{i}\left(v_{l}\right)+e_{k}\left(v_{l}\right)$ does not depend on $X$ therefore we can write $E_{k}(X)=C P_{l}-e_{k}(X)$ for $X \in\left(d_{l}, v_{l}\right]$. By induction hypothesis, we have:

$$
E_{m}(X)=H_{i}^{m}-e_{j_{i}^{m}}^{m}(X), \quad m \in S_{k} \text { where } j_{i}^{m} \in C P_{i}
$$

Therefore,

$$
\begin{aligned}
\sum_{m \in S_{k}} E_{m}(X) & =\sum_{m \in S_{k}} H_{i}^{m}-\sum_{m \in S_{k}} e_{j_{i}^{m}}(X) \\
& =H_{i}-e_{j_{i}}(X)
\end{aligned}
$$

where $j_{i} \in C P_{k}$ and $H_{i}=\sum_{m \epsilon S_{k}} H_{i}^{m}$. Therefore, the result holds for $E_{k}(x)$. Again, the Algorithm 1 holds, including the following extra steps:

Algorithm 3: Given $\left(j_{1}^{m}, \ldots \ldots, j_{I^{m}}^{m}\right),\left(h_{1}^{m}, \ldots \ldots, h_{I^{m}}^{m}\right)$ and $H_{1}^{m}$ corresponding to $E_{m}$ for every $m \in S_{k}$.

* Step 0: Sort the breakpoints $h_{1}^{m}, \ldots \ldots, h_{I^{m}}^{m}$ for $m \in S_{k}$ in increasing order and rename them ( $h_{1}, \ldots \ldots ., h_{I}$ ) where $I=\sum_{m \in S_{k}} I^{m}$. Construct the corresponding vector $\left(j_{1}, \ldots \ldots ., j_{I}\right)$ such that $j_{i}$ is a composite product involving product $j_{l}^{m}$ iff $h_{l-1}^{m}<h_{k} \leq h_{l}^{m}$ for $l=1, \ldots \ldots, I^{m}$ and $m \in S_{k}$
* Steps 1 to 4: See Algorithm 1.

In this case, the value function is also convex and that the complexity of Algorithm 3 is also $O\left(n^{2}\right)$.

## NUMERICAL RESULTS

We suggests two heuristics to solve the assortment planning problem under nested preferences with random proportion, i.e., the problem formulated in section4.3. We numerically evaluate the performance of these heuristics with respect to the upper bound established in Proposition 5.
We also benchmark their performance against two previously known heuristics, one based on static substitution, and the other being the Sample Path Gradient Algorithm of Mahajan and van Ryzin [1].

## Srivastava and Srivastava

Our first heuristic is known as the fixed proportion heuristic (NF). In this heuristic, the assortment planning problem solved under nested preferences using the same parameters but with fixed proportion, and the corresponding optimal inventory levels in the assortment planning problem with random proportion are used.
Let $x^{* N F}$ be the vector of inventory levels in this solution. Our second heuristic is known as the modified fixed proportion heuristic (MNF). Again consider the optimal solution to the assortment planning problem under fixed proportion.

Let $K=\left\{k_{1}, \ldots \ldots, k_{s}\right\} \subseteq\{1, \ldots \ldots, r\}$ be the set of all products such that $x_{k}^{* N F}>0$ and let $\left(\emptyset_{k_{1}, k_{2}}, \ldots \ldots ., \emptyset_{k_{s-1}, k_{s}}, \emptyset_{k_{s}}\right)$ be the set of optimal critical fractiles as defined after Lemma 3. In the MNF heuristic, we use the same critical fractiles as in the NF heuristic, but we obtain inventory levels, $x^{* M N F}$, from the true distribution of demand based on random proportions. For ease of computation, we estimate the true distribution of demand by simulation. This estimation is done sequentially for all products in the order of the nested preferences.
Let $\tilde{G}_{k_{m}}$ denote the estimated cdf of demand for product $k_{m}$ given inventory levels $x_{k_{1}}^{* M N F}, \ldots \ldots, x_{k_{m-1}}^{* M N F}$ for products $j_{1}, \ldots \ldots, j_{m-1}$. Then, we have

$$
x_{k_{m}}^{* M N F}= \begin{cases}G_{k_{m}}^{-1}\left(\emptyset_{k_{m}}, k_{m+1}\right) & m=1, \ldots \ldots, s-1 \\ G_{k_{m}}^{-1}\left(\emptyset_{k_{m}}\right) & m=s\end{cases}
$$

In the static substitution, the optimal solution is to carry only one product in the assortment, i.e. the one that yields the highest expected profit when stocked alone. This gives the solution under the static substitution heuristic ( S ) and it is denoted as $x^{* S}$.
Finally, the solution under the Sample Path Gradient Heuristic (SPGA) of Mahajan and van Ryzin (12001) is denoted as $x^{* S P}$. We vary the cost and price parameters of the products, the mean demand, and the proportions of customers who are willing to buy each product in order to evaluate the performance of the heuristics in different cases. The parameters in all problem instances are such that the following conditions are satisfied:

$$
u_{k}+o_{k} \geq \alpha_{k+1}\left(u_{k+1}+o_{k+1}\right) \quad \text { for } k=1, \ldots \ldots, r-1
$$

This implies that condition (23) is satisfied for every inventory vector, and therefore, by Proposition $5, P_{E} \Pi^{N F}\left(x^{* N F}\right)$ constitutes an upper bound on the performance of the four heuristics. In addition, we choose all parameters in such a way that all $n$ products are stocked in the optimal solution in the NF heuristic. Our numerical study is based on a potential assortment of five products.

We generate demand $D$ for the product category using a Poisson random variable with mean $\mu$. For each customer, we generate the customer type using a multinomial distribution with parameters $\left(\delta_{1}, \ldots \ldots, \delta_{r}\right) ;$ where $\delta_{k}$ is the probability that the customer is willing to buy products 1 to $k$. Given the inventory levels from the four heuristics, we use the distribution of demand to estimate the expected profits under all four heuristics as well as under the upper bound. This estimation is done by simulation using a common set of 10,000 sample paths of random numbers in all cases. Let $P_{E} \Pi^{N R}(x)$ denote the expected profit for the assortment planning problem under random proportion for an inventory vector $x$. The optimality gaps (OG) of the four heuristics with respect to the upper bound are computed as:

$$
\frac{P_{E} \Pi^{N F}\left(x^{* N F}\right)-P_{E} \Pi^{N R}(x)}{P_{E} \Pi^{N F}\left(x^{* N F}\right)} ; \text { where } x \text { is se equal to } x^{* N F}, x^{* M N F}, x^{* S} \& x^{* S P}
$$

The inventory levels in the four heuristics are computed in the following described manner. Since the NF heuristic requires a continuous distribution, therefore we need to use the normal approximation of the Poisson distribution to compute $x^{* N F}$. We also use this same approximation in the S heuristic to compute $x^{* S S}$. For the MNF heuristic, we use the Poisson and multinomial distributions to compute $x^{* M N F}$ given ( $\emptyset_{k_{1}, k_{2}}, \ldots \ldots, \emptyset_{k_{s-1}, k_{s}}, \emptyset_{k_{s}}$ ) obtained from the NF heuristic.

Finally, for SPGA, we use the Poisson and multinomial distributions to compute $x^{* S P}$. Since SPGA is a simulation-based algorithm, we use the following parameters for its implementation: number of iterations $=10,000$, starting inventory vector, $x_{k}^{0}=\mu \gamma_{k} / r$ and step size $=1 /$ iteration index. The step
size values chosen by us are the same as used by Mahajan and van Ryzin (12001). We note that the inventory levels are discrete for the MNF heuristic, but are real numbers for all other heuristics. For our purposes, we do not round the inventory levels to the nearest integers.
We present the results of the numerical study by grouping the problem instances into three scenarios, based on which variable is varied while everything else is kept constant.
In Scenario 1, we study the impact of the $\gamma_{k}$ parameters on the performance of the heuristics.
In Scenario 2, we study the impact of the amount of safety stock.
Finally, in Scenario 3, we study the impact of mean demand.
Table 3: Underage and overage cost parameters

|  | $\mathbf{P 1}$ | $\mathbf{P 2}$ | $\mathbf{P 3}$ | $\mathbf{P 4}$ | $\mathbf{P 5}$ |
| :--- | :--- | :--- | :--- | :--- | :--- |
| $\mathbf{u}$ | 15 | 14 | 13 | 12 | 11 |
| $\mathbf{0}$ | 10.15 | 6.15 | 4.29 | 3.29 | 2.75 |

In Scenario 1, we fix the underage and overage parameters as shown in Table 3 and set mean demand $\mu$ to 30. The corresponding optimal critical fractiles are: $\left(\emptyset_{1,2}, \ldots \ldots ., \emptyset_{4,5}, \emptyset_{5}\right)=$ ( $0.2,0.35,0.5,0.65,0.8$ )

We generate 11 problem instances by varying the proportions $\gamma_{k}$ in such a way that $\sum_{k=1}^{r} \gamma_{k}$ decreases from 5 to 3 in steps of 0.2 . When $\sum_{k=1}^{r} \gamma_{k}=5$ all customers are willing to buy all 5 products, i.e. this is the homogeneous population case. As the value of this sum decreases, the proportion of customers willing to buy the least preferred products decreases and eventually when the value reaches 3 , all customers are equally likely of being of type (1), (1,2), $(1,2,3),(1,2,3,4)$ or (1,2,3,4,5).

| $\gamma$ | NF |  | MNF |  | S Substitution |  | SP |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | $\mathbf{x}$ | $\begin{aligned} & \text { OG(\% } \\ & \text { ) } \\ & \hline \end{aligned}$ | x | $\begin{aligned} & \text { OG(\% } \\ & \hline \end{aligned}$ | x | $\begin{aligned} & \text { OG(\% } \\ & \text { ) } \end{aligned}$ | X | $\begin{aligned} & \text { OG(\% } \\ & \hline \end{aligned}$ |
| (1,1,1,1) | (44.0,3.2,2.7,2.7,3.2 | 0.00 | $(44,3,3,3,3$ | 0.00 | $\begin{aligned} & \text { (51.7,0,0,0,0 } \\ & \text { ) } \end{aligned}$ | 2.13 | (25.2,14.5,8.4,4.0,2.6) | 2.46 |
| (1,1,1,1,0.8) | $\begin{aligned} & \text { (44.0,3.2,2.7,4.7,1.0 } \\ & \hline \end{aligned}$ | 0.01 | $\begin{aligned} & (44,3,3,5,1 \\ & \hline \end{aligned}$ | 0.00 | $\begin{aligned} & \text { (51.7,0,0,0,0 } \\ & \text { ) } \end{aligned}$ | 2.10 | (25.5,16.7,8.7,5.0,0.6) | 2.40 |
| (1,1,1,0.8,0.8) | $\begin{aligned} & \text { (44.0,3.2,5.8,0.0,2.3 } \\ & \hline \end{aligned}$ | 0.03 | $\begin{aligned} & (44,3,6,0,2 \\ & ) \end{aligned}$ | 0.02 | $\begin{aligned} & \text { (51.7,0,0,0,0 } \\ & \text { ) } \end{aligned}$ | 1.99 | $\begin{aligned} & \text { (26.5,17.8,10.7,0.0,0.9 } \\ & \hline \end{aligned}$ | 2.13 |
| (1,1,0.8,0.8,0.8) | $\begin{aligned} & \text { (44.0,6.9,0.0,1.4,2.6 } \\ & \text { ) } \end{aligned}$ | 0.03 | $\begin{aligned} & \text { (44,7,0,1,3 } \\ & \text { ) } \\ & \hline \end{aligned}$ | 0.03 | $\begin{aligned} & (51.7,0,0,0,0 \\ & ) \end{aligned}$ | 1.74 | (30.4,21.6,.0,0.9,2.0) | 1.47 |
| $\begin{aligned} & (1,0.8,0.8,0.8,0.8 \\ & ) \\ & \hline \end{aligned}$ | $\begin{aligned} & \text { (48.3,0.0,1.3,2.2,2.6 } \end{aligned}$ | 0.07 | $\begin{aligned} & \text { (48,0,2,2,2 } \\ & \hline \end{aligned}$ | 0.07 | $\begin{aligned} & \text { (51.7,0,0,0,0 } \\ & \text { ) } \end{aligned}$ | 1.23 | (40.8,7.3,2.8,0.8,1.3) | 2.16 |
| (1,0.8,0.8,0.8,0.6 | (48.3,0.0,1.3,3.9,0.7 | 0.08 | $\begin{aligned} & (48,0,2,3,1 \\ & \hline \end{aligned}$ | 0.08 | $\begin{aligned} & \text { (51.7,0,0,0,0 } \\ & \hline \end{aligned}$ | 1.20 | (40.9,7.4,3.1,1.5,0.0) | 2.15 |
| $\begin{aligned} & \text { (1,0.8,0.8,0.6,0.6 } \\ & \text { ( } \\ & \hline \end{aligned}$ | $\begin{aligned} & \text { (48.3,0.0,4.0,0.0,1.6 } \\ & \text { ) } \\ & \hline \end{aligned}$ | 0.10 | $\begin{aligned} & \text { (48,0,4,0,2 } \\ & \text { ) } \\ & \hline \end{aligned}$ | 0.10 | $\begin{aligned} & \text { (51.7,0,0,0,0 } \\ & \text { ) } \end{aligned}$ | 1.09 | (39.8,8.2,4.9,0.0,0.0) | 2.68 |
| $\begin{aligned} & (1,0.8,0.6,0.6,0.6 \\ & )^{\prime} \end{aligned}$ | (48.6,2.2,0.0,0.8,1.9 | 0.11 | $\begin{aligned} & (49,2,0,1,2 \\ & )^{2} \end{aligned}$ | 0.10 | $\begin{aligned} & \text { (51.7,0,0,0,0 } \\ & \text { ) } \end{aligned}$ | 0.87 | (42.1,10.0,0.0,0.0,0.8) | 1.78 |
| $\begin{aligned} & (1,0.8,0.6,0.6,0.4 \\ & )^{( } \\ & \hline \end{aligned}$ | (48.6,2.2,0.0,2.2,0.4 | 0.13 | $\begin{aligned} & \text { (49,2,0,2,0 } \\ & \text { ( } \end{aligned}$ | 0.10 | $\begin{aligned} & \text { (51.7,0,0,0,0 } \\ & \text { ) } \end{aligned}$ | 0.85 | (41.9,10.1,0.0,0.0,0.0) | 1.87 |
| $\begin{aligned} & (1,0.8,0.6,0.4,0.4 \\ & )^{( } \\ & \hline \end{aligned}$ | $\begin{aligned} & \text { (48.6,2.4,1.2,0.0,0.9 } \\ & \text { ) } \end{aligned}$ | 0.13 | $\begin{aligned} & \text { (49,0,1,0,1 } \\ & \mathbf{r}^{2} \end{aligned}$ | 0.12 | $\begin{aligned} & \text { (51.7,0,0,0,0 } \\ & \text { ) } \end{aligned}$ | 0.77 | (40.7,11.3,0.0,0.0,0.0) | 2.34 |
| $\begin{aligned} & (1,0.8,0.6,0.4,0.2 \\ & ) \\ & \hline \end{aligned}$ | $\begin{aligned} & \text { (48.6,2.4,1.3,0.6,0.1 } \\ & \text { ) } \end{aligned}$ | 0.14 | $\begin{aligned} & \text { (49,2,1,1,0 } \\ & \hline \end{aligned}$ | 0.12 | $\begin{aligned} & \text { (51.7,0,0,0,0 } \\ & \text { ) } \end{aligned}$ | 0.76 | (40.7,11.3,0.0,0.0,0.0) | 2.32 |
| mean |  | 0.08 |  | 0.07 |  | 1.34 |  | 2.16 |

Table 4 - Scenario 1
Table 4 reports the value of the optimal quantity under all 4 heuristics as well as their optimality gaps with respect to the upper bound. We see that as $\sum_{k=1}^{r} \gamma_{k}$ increases, the optimality gap of the NF heuristic decreases from $0.14 \%$ to $0 \%$, and that of the MNF heuristic decreases from $0.12 \%$ to $0 \%$. The MNF heuristic performs better than the NF heuristic in all cases. The optimality gaps 28 of the S and SPGA heuristics are both much larger than those of the NF and MNF heuristics.

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Further, the optimality gap of the SS heuristic behaves differently from the NF and MNF heuristics because it decreases as $\sum_{k=1}^{r} \gamma_{k}$ decreases; however, it remains larger in all cases. In Scenario 2, we keep the same values for the underage costs of each product (see Table 3), but vary the overage cost vector so as to have the optimal critical fractiles be equidistant with $\emptyset_{12}=0.1$ and $\emptyset_{5}$ which varies between 0.2 and 0.9 . This has the effect of varying the total amount of safety stock. We also set mean demand $\mu$ to 50 and $\delta_{k}=0.7$ for $k=2, \ldots \ldots ., 5$.

Table 5- Scenario 2

| $\emptyset_{r}$ | NF |  | MNF |  | S Substitution |  | SP |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | x | $\begin{aligned} & \text { OG(\% } \\ & \text { ) } \\ & \hline \end{aligned}$ | x | $\begin{aligned} & \hline \text { OG(\% } \\ & \text { ) } \end{aligned}$ | x | $\begin{aligned} & \hline \text { OG(\% } \\ & \text { ) } \end{aligned}$ | x | $\begin{aligned} & \text { OG(\% } \\ & \mathbf{l}^{2} \\ & \hline \end{aligned}$ |
| 0.2 | (43,0.3,0.1,0.1,0.0) | 0.25 | $\begin{aligned} & \text { (43,0,0,0,0 } \\ & )^{2} \end{aligned}$ | 0.03 | $\begin{aligned} & \text { (43.4,0,0,0,0 } \\ & \mathbf{r}^{2} \\ & \hline \end{aligned}$ | 0.08 | $\begin{aligned} & \text { (40.0,2.8,0.0,0.0,0.0 } \end{aligned}$ | 1.07 |
| 0.3 | $\begin{aligned} & \hline \text { (44.1,0.6,0.3,0.1,0.1 } \\ & \hline \end{aligned}$ | 0.27 | (44,1,0,0,0 | 0.10 | $(45.0,0,0,0,0$ | 0.14 | (42.1,2.6,0.0,0.0,0.0 | 0.58 |
| 0.4 | (44.9,1.0,0.4,0.2,0.1 | 0.27 | $(45,1,0,0,0$ | 0.15 | (46.3,0,0,0,0 | 0.31 | $(43.6,2.4,0.0,0.0,0.0$ | 0.32 |
| 0.5 | (45.5,1.5,0.6,0.3,0.1 | 0.24 | $(45,2,0,0,0$ | 0.21 | (47.3,0,0,0,0 | 0.52 | (43.3,3.7,0.0,0.0,0.0 | 0.55 |
| 0.6 | (46.0,1.9,0.7,0.3,0.2 | 0.24 | (46,2,0,0,0 | 0.30 | $(48.2,0,0,0,0$ | 0.81 | (43.0,5.1,0.0,0.0,0.0 | 0.83 |
| 0.7 | (46.3,2.4,0.9,0.4,0.2 | 0.23 | (46,2,1,0,0 | 0.29 | $(48.9,0,0,0,0$ | 1.11 | (44.4,5.2,0.1,0.1,0.0 | 0.56 |
| 0.8 | (46.6,3.0,1.1,0.6,0.3 | 0.21 | (47,3,1,0,0 | 0.21 | (49.7,0,0,0,0 | 1.50 | (45.3,5.1,0.2,0.4,0.0 | 0.38 |
| 0.9 | (46.9,3.5,1.4,0.8,0.5 | 0.18 | $(47,3,2,0,1$ | 0.18 | (50.3,0,0,0,0 | 1.95 | $(45.2,5.9,1.0,0.0,1.0$ | 0.41 |
| $\begin{aligned} & \hline \text { mea } \\ & \mathrm{n} \\ & \hline \end{aligned}$ |  | 0.24 |  | 0.18 |  | 0.80 |  | 0.59 |

In Table 5, we see that the performance of the S Substitution heuristic deteriorates as the amount of safety stock increases while the optimality gap of the other heuristics are fairly constant. We also see that the MNF heuristic significantly improves on the NF heuristics and that both always do better than the SPGA. Note that as the amount of safety stock increases, the S Substitution heuristic increases the inventory level of the first product only, while the remaining three heuristics increase inventory levels of all products, with total inventories of products 2-5 increasing more than the inventory of product 1.Thus, the S Substitution heuristic is unable to exploit the differences between critical fractiles of products, which the other heuristics are able to. In Scenario 3, we use the same cost parameters as in Table 3 and set $\delta_{k}=\frac{1}{5}$ for $k=1, \ldots \ldots ., r$
so that the customers are equally likely to be of each possible type. We vary mean demand $\mu$ between 30 and 110 by steps of 20 .

Table 6- Scenario 3

| $\boldsymbol{\mu} \boldsymbol{\mu}$ | $\mathbf{N F}$ | $\mathbf{M N F}$ |  | SSubstitution |  | $\mathbf{S P}$ |  |  |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
|  | $\mathbf{x}$ | $\mathbf{O G ( \% )}$ | $\mathbf{x}$ | $\mathbf{O G ( \% )}$ | $\mathbf{x}$ | $\mathbf{O G ( \% )}$ | $\mathbf{x}$ | $\mathbf{O G ( \% )}$ |
| 10 | $(9.4,1.1,0.6,0.3,0.1)$ | 0.84 | $(9,1,1,0,0)$ | 0.67 | $(10.8,0,0,0,0)$ | 1.89 | $(9.3,1.4,0.5,0.0,0)$ | 0.76 |
| 30 | $(28.9,1.9,1.0,0.5,0.1)$ | 0.27 | $(29,2,1,0,0)$ | 0.20 | $(31.3,0,0,0,0)$ | 1.02 | $(27.9,3.8,0.4,0.0,0)$ | 0.39 |
| 50 | $(48.6,2.4,103,0.6,0.1)$ | 0.15 | $(49,2,1,1,0)$ | 0.13 | $(51.7,0,0,0,0)$ | 0.74 | $(39.8,12.2,0.0,0.0,0)$ | 2.88 |
| 70 | $(68.3,2.9,1.5,0.7,0.2)$ | 0.11 | $(68,3,1,1,0)$ | 0.10 | $(72.0,0,0,0,0)$ | 0.61 | $(51.1,19.1,0.8,0,0)$ | 4.66 |
| 90 | $(88.1,3.3,1.7,0.8,0.2)$ | 0.08 | $(88,3,2,1,0)$ | 0.07 | $(92.3,0,0,0,0)$ | 0.57 | $(60.4,28.3,1.3,0,0)$ | 6.56 |
| 110 | $(107.9,3.6,1.9,0.9,0.2)$ | 0.07 | $(108,3,2,1,0)$ | 0.07 | $(112.6,0,0,0,0)$ | 0.49 | $(36.9,36.0,6.1,0,0)$ | 8.81 |
| 130 | $(127.7,3.9,2.1,1.0,0.2)$ | 0.04 | $(128,4,2,1,0)$ | 0.03 | $(132.8,0,0,0,0)$ | 0.45 | $(69.7,42.5,15.5,0,2.0)$ | 11.47 |
| mean |  | 0.22 |  | 0.18 |  | 0.82 |  |  |
| 5 |  |  |  |  |  |  |  |  |

In Table 6, as mean demand increases, the optimality gaps of the NF, MNF and S Substitution heuristics decrease while that of the SPGA increases. Also the NF and MNF heuristics perform very well, with an average optimality gap of $0.18 \%$ and $0.22 \%$ respectively. The poor performance of the SPGA indicates that the algorithm may need more iteration to converge for a larger mean demand than for a smaller mean demand. In total (including the 3 scenarios) we generated 230 problem instances.

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The average optimality gaps of the heuristics were equal to $0.18 \%$ and $0.14 \%$ respectively for the NF and MNF heuristics, compared to $0.54 \%$ and $1.57 \%$ for the SPGA and SS heuristics.
Finally, we come to know that assuming static substitution can lead to a substantial loss in expected profit when customers actually dynamically substitute, in particular, when a large amount of safety stock is required, and when the total proportion of customers buying each product is high.
It is also observed that the NF heuristic performs well, especially when mean demand and the proportion of customers willing to substitute are high. The heuristic often does better than the SPGA, especially when mean demand is large. Moreover, it significantly reduces the computing time because it is not simulation-based but instead uses an efficient DP algorithm.
Finally, in our results, the MNF heuristic performs better than the NF heuristic in all cases. This may not be true in general, and a decision-maker may consider computing both heuristics and taking the higher value between them. Note that the MNF heuristic is slightly more computationally intensive because one needs to compute the true distribution of demand under random proportion. It remains, however, significantly faster than the SPGA, given the parameters that we chose.
While our heuristics perform better than the SPGA for the chosen preference structure, it should be noted that SPGA is a very general algorithm which can handle any type of preference structure whereas the NF and MPF heuristics can only deal with nested (and outtree-like) preferences.
Moreover it offers the guarantee of converging to a stationary point of the expected profit function, while our heuristics do not. In contrast, our NF heuristic presents the advantage of being optimal in one setting, namely, the homogeneous population case.

## CONCLUSION

The optimal assortment and inventory levels are obtained by us under dynamic substitution when customers have homogeneous, nested and outtree-shaped preferences with trend-following or fixed partitioning of demand. The dynamic program algorithm used to compute the solution involves the maximization of the sum of a concave and a convex function which may lead to points of nondifferentiability in the value function. Also, we were able to bound the number of breakpoints in the value function and prove that the complexity of the algorithm is $O\left(n^{2}\right)$. Under nested preferences with random partitioning of demand, we show that the algorithm provides a good heuristic which gives results similar or better than those obtained with the SPGA, and significantly reduces computation time. The key managerial insights of our paper are as follows. Firstly, we showed that ignoring dynamic substitution can lead to a substantial loss in expected profit because the retailer does not take advantage of the differences in return and risk among products.
Secondly, we showed that inventory cost economics should be considered, along with customer heterogeneity and competition, as a driver of product variety. This insight is relevant for applications in product design and pricing problems under competition.
And finally, we showed that contrary to previous research under static substitution, it is not necessarily optimal to stock the most preferred or the most profitable product in the assortment.
The next steps of this research consist in studying the case of acyclic and general preferences using an approach similar to the one we used for nested and outtree-shaped preferences.
Our results are derived under the assumption that the prices of the products are exogenous. An interesting extension of our work would be to consider prices are decision variables which would influence not only profit but also the proportions of customers willing to buy each product.

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## Appendix

## Proof of Lemma 3.4

Proof: (by induction) Consider $E_{r}$, then from eqn. (15), we have:

$$
E_{r}^{\prime}\left(G^{-1}\left(\emptyset_{r}\right)^{-}\right)=e_{r}^{\prime}\left(G^{-1}\left(\emptyset_{r}\right)\right)=0=E_{r}^{\prime}\left(G^{-1}\left(\emptyset_{r}\right)^{+}\right)
$$

Considering that the result is true for $E_{k+1}$. We know that $e_{k}(x)$ is differentiable everywhere. Therefore, the points where $Z_{k}(x)=E_{k+1}(x)+e_{k}(x)$ is not differentiable are breakpoints $h_{m}$ of $E_{k+1}$
such that $E_{k+1}^{\prime}\left(h_{m}^{-}\right)<E_{k+1}^{\prime}\left(h_{m}^{+}\right)$.
Similarly from eqn. (12), $Z_{k}^{\prime}\left(h_{m}^{-}\right)<Z_{k}^{\prime}\left(h_{m}^{+}\right)$.
These points $h_{m}$ cannot be local maxima of $Z_{k}$ because the right derivative is greater than the left derivative.
It follows that if $v$ is a local maximum of $Z_{k}$ then $Z_{k}^{\prime}(v)=0$. Therefore in (16),
$Z_{k}^{\prime}\left(v_{l}\right)=0$ for $l=1, \ldots \ldots$, s and $Z_{k}^{\prime}\left(v_{l}^{-}\right)=Z_{k}^{\prime}\left(v_{l}^{+}\right)=0$ which gives $E_{k}^{\prime}\left(v_{l}^{-}\right)=E_{k}^{\prime}\left(v_{l}^{+}\right)$.
From Proposition 1, $Z_{k}$ is decreasing at $d_{l}, l=1, \ldots \ldots, s$ and $\xi_{k}$ is constant in the segments $\left(d_{l}, v_{l}\right) ; l=1, \ldots \ldots, s$.
Therefore we have $\xi_{k}^{\prime}\left(d_{l}^{-}\right)<\xi_{k}^{\prime}\left(d_{l}^{+}\right)=0$, which gives $E_{k}^{\prime}\left(d_{l}^{-}\right)<E_{k}^{\prime}\left(d_{l}^{+}\right)$.
It is mentioned in Proposition 1, the points $v_{i}$ and $d_{i}$ for $i=1, \ldots \ldots, s$ are the breakpoints of $E_{k}$ that are not breakpoints of $E_{k+1}$, therefore, the result is true for $E_{k}$.

## Proof of Proposition 2

Proof. We know that $E_{k}$ is convex between breakpoints as $e_{k}$ is concave for every $k$ [from eqn. (14)]. Let $h_{i_{1}}$ and $h_{i_{2}}$ be the first two NDBP of $E_{k}$ (if there are less than two then the proof is simpler). By convexity of $E_{k}$ and continuity of its derivative in $\left[0, h_{i_{1}}\right)$;
$E_{k}(x) \geq E_{k}\left(h_{i_{1}}\right)+\left(x-h_{i_{1}}\right) E_{k}^{\prime}\left(h_{i_{1}}^{-}\right)$for $x \in\left[0, h_{i_{1}}\right)$
Similarly we get,
$E_{k}(x) \geq E_{k}\left(h_{i_{1}}\right)+\left(x-h_{i_{1}}\right) E_{k}^{\prime}\left(h_{i_{1}}^{+}\right)$for $x \in\left[h_{i_{1}}, h_{i_{2}}\right)$
By Lemma 10 along with (24) and (25) implies that:

$$
\begin{equation*}
E_{k}(x) \geq E_{k}\left(h_{i_{1}}\right)+\left(x-h_{i_{1}}\right) E_{k}^{\prime}\left(h_{i_{1}}^{-}\right) \text {for } x \in\left[0, h_{i_{2}}\right) \tag{25}
\end{equation*}
$$

Then from induction, all subgradients of $E_{k}$ lie below $E_{k}$ and therefore the curve is strictly convex in [ $0, h_{i}$ ]. By Lemma (3.4), at the last breakpoint $h_{i}$ we have:

$$
E_{k}^{\prime}\left(h_{I}^{-}\right) \leq E_{k}^{\prime}\left(h_{I}^{+}\right)=0
$$

Since $E_{k}$ is strictly convex in $\left[0, h_{I}\right], E_{k}^{\prime}$ is negative for $X \leq h_{I}$ so that $E_{k}$ is decreasing.
Proof of Lemma 4: We first need to establish the following Lemma:
Lemma 10. For $E_{k}$ given by (14), $e_{j_{i}}^{\prime}(x) \leq-E_{k}^{\prime}\left(x^{-}\right)$for $x \leq h_{i} ; i=1, \ldots \ldots . I$
Proof. By Lemma 3.4, we have $e_{j_{i}}^{\prime}\left(h_{i-1}\right) \leq-E_{k}^{\prime}\left(h_{i-1}{ }^{-}\right) \leq-E_{k}^{\prime}\left(h_{i-1}{ }^{+}\right)$
Assume that we have $e_{j_{i}}^{\prime}(x) \leq-E_{k}^{\prime}\left(x^{-}\right)$for $x \in\left(h_{m}, h_{i}\right]$
But (contradiction) that $e_{j_{i}}^{\prime}(x) \leq E_{k}^{\prime}\left(x^{-}\right)$for $x \in\left(h_{m-1}, h_{m}\right]$

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This, along with (26), imply that there exists $h_{m-1}<x_{1} \leq h_{m}$ such that $e_{j_{i}}^{\prime}\left(x_{1}\right) \leq-E_{k}^{\prime}\left(x_{1}\right)$. If $h_{m}$, is an NDBP, we have $e_{j_{i}}^{\prime}\left(h_{m}\right) \leq-E_{k}^{\prime}\left(h_{m}{ }^{+}\right)$. By Lemma 3.4, we have $-E_{k}\left(h_{m}{ }^{-}\right)>-E_{k}\left(h_{m}{ }^{+}\right)$. So that $e_{j_{i}}^{\prime}\left(h_{m}\right) \leq-E_{k}^{\prime}\left(h_{m}{ }^{-}\right)$and therefore, there exist $x_{1}$ in the mentioned manner.
Since, $-E_{k}^{\prime}\left(x_{1}\right)=e_{j_{m}}^{\prime}\left(x_{1}\right)$ and $e_{j_{i}}^{\prime}$ and $e_{j_{m}}^{\prime}$ can only cross once by the at-most-one-time-crossing property, we have:

$$
\begin{aligned}
& e_{j_{i}^{\prime}}^{\prime}(x)>e_{j_{m}}^{\prime}(x) \text { for } x<x_{1} \\
& e_{j_{i}}^{\prime}(x)<e_{j_{m}}^{\prime}(x) \text { for } x>x_{1}
\end{aligned}
$$

There are two cases. First assume that $j_{i}<j_{m}$.
In this case $Z_{j_{i}}^{\prime}(x)=e_{j_{i}}^{\prime}(x)-e_{j_{m}}^{\prime}(x) \geq 0$ for $x \in\left(h_{m-1}, x_{1}\right)$. From Proposition 1, this should imply that $E_{j_{i}}^{\prime}(x)=e_{j_{i}}^{\prime}(x)$ in that interval. However, also there exist $E_{k}^{\prime}(x)=e_{j_{m}}^{\prime}(x)$ in that interval. Since $\leq j_{i}<j_{m}$, we have the index of the $e$ function at $x$ increases when going from $E_{j_{i}}$ to $E_{k}$ and this violates the non-increasing index property. So we have a contradiction. Let us consider $j_{m}<j_{i}$. In this case, $Z_{j_{m}}^{\prime}(x)=e_{j_{m}}^{\prime}(x)-e_{j_{i}}^{\prime}(x) \geq 0$ for $x \in\left(h_{i-1}, h_{i}\right)$.
But from Proposition 1, we have $E_{j_{m}}^{\prime}(x)=e_{j_{m}}^{\prime}(x)$ in that interval. However, also there exist $E_{k}^{\prime}(x)=e_{j_{i}}^{\prime}(x)$ in that interval. Since $\leq j_{m}<j_{i}$, we have the index of the $e$ function at $x$ increases when going from $E_{j_{m}}$ to $E_{k}$ and this violates the non-increasing index property. So we have a contradiction. Now, returning back to the proof of lemma 4.


Proof. Considering the index $i$ as a "repetition" if in (14) there exists $j_{m}=j_{i}$ and $m<i$. Otherwise $i$ is an "original". Choosing $k$ be a repetition and let $y$ be the unique index such that $j_{i}=$ $j_{y}, y<i$ and $j_{m} \neq j_{i}$ for $m=y+1, \ldots \ldots ., i-1$ Now, choosing $y+1$ as an original, then we have $y<i-1$ as there should be at least one other product between two occurrences of the same product. From Lemma 10, we have

$$
\begin{aligned}
& e_{j_{y+1}^{\prime}}^{\prime}(x)<e_{j_{y}}^{\prime}(x)=-E_{k}^{\prime}(x) \text { for } x \in\left(h_{y-1}, h_{y}\right) \\
& e_{j_{i}}^{\prime}(x)<e_{j_{y+1}}^{\prime}(x)=-E_{k}^{\prime}(x) \text { for } x \in\left(h_{y}, h_{y+1}\right)
\end{aligned}
$$

Since $j_{i}=j_{y}$, therefore we have, $e_{j_{y}}^{\prime}\left(h_{y}\right)=e_{j_{i}}^{\prime}\left(h_{y}\right)=e_{j_{y+1}}^{\prime}\left(h_{y}\right)$. The at-most-one-time-crossing property shows that: $e_{j_{y+h}}^{\prime}(x)<e_{j_{y}}^{\prime}(x)$ for $x<h_{y}$. We cannot have $j_{l}=j_{y+1}$ with $l<y$ because this would imply that $e_{j_{l}}^{\prime}(x)=e_{j_{l+1}}^{\prime}(x)=-E_{k}^{\prime}(x)<e_{j_{y}}^{\prime}(x)$ for $x \in\left(h_{l-1}, h_{l}\right)$ and this contradicts Lemma 10. This proves that $y+1$ is an original. We say that original $y+1$ is associated to repetition . Let $v$ be the number of originals. Let $w$ be the number of repetitions. The number of breakpoints in $E_{k}$ is given by $v+w$. To find the maximum number of breakpoints we solve the following optimization problem:
$\max$
such that

$$
\begin{aligned}
& \mathrm{V}+\mathrm{W} \\
& \mathrm{~V} \leq \mathrm{r}-\mathrm{k}+1 \\
& \mathrm{~W} \leq \mathrm{V}-1
\end{aligned}
$$

The first constraint comes from the fact that only products $\{k, \ldots \ldots, r\}$ can appear in $E_{k}$. The second constraint comes from the above claim as for each repetition, there exist an original associated with it and this original is located on the right of the previous occurrence of that product. This implies that each original can be associated to at most one repetition, except for the first index which cannot be associated with any repetition. The number of breakpoints of $E_{k}$ is maximized when $v=r-k+$ 1 and $w+1=v$ such that $v+w=2(r-k)+1$.

