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Original Article

Gamma Vector Spaces and their Generalization

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ABSTRACT
In this paper, we have developed some characterizations of gamma vector spaces and proved that the set of all linear gamma transformation forms a gamma ring. Our results are the genetralizations of that of Hiram Paley and Paul M. Weichsel[2].
KEYWORDS: *Gamma, Ring, Homomorphism, Generalization, Transformation.*

INTRODUCTION

N. Nobusawa [5] introduced the concept of a Γ-ring which is called the Γ-ring in the sense of Nobusawa. He obtained an analogue of the Wedderburn's Theorem for Γ-rings with minimum condition on left ideals. W. E. Barnes [1] gave the definition of a Γ-ring as a generalization of a ring and he also developed some other concepts of Γ-rings such as Γ-homomorphism, prime and primary ideals, m-systems etc. Hiram Paley and Paul M. Weichsel [2] studied classical vector spaces. Here they also developed a number of remarkable results in ring theories.

In this paper, we consider the Γ-rings due to Barnes and study the analogous results of Hiram Paley and Paul M. Weichsel [2] in Γ-rings. We also obtain the Wedderburn's Theorem in Γ-rings which is the generalization of that in [2].

1. PRELIMINARIES

1.1. Definitions:

Gamma Ring: Let M and Γ be two additive abelian groups. Suppose that there is a mapping from $M \times \Gamma \times M \to M$ (sending (x, α, y) into $x\alpha y$) such that

i) $(x + y)\alpha z = x\alpha z + y\alpha z$
ii) $x(\alpha + \beta)z = x\alpha z + x\beta z$
iii) $x\alpha(y + z) = x\alpha y + x\alpha z$
iv) $(x\alpha y)\beta z = x\alpha(y\beta z)$,

where $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. Then M is called a Γ-ring.

Ideal of Γ-rings: A subset A of the Γ-ring M is a left (right) ideal of M if A is an additive subgroup of M and $M\Gamma A = \{c\alpha a \mid c \in M, \alpha \in \Gamma, a \in A\}(A\Gamma M)$ is contained in A. If A is both a left and a right ideal of M, then we say that A is an ideal or two sided ideal of M.

If A and B are both left (respectively right or two sided) ideals of M, then $A + B = \{a + b \mid a \in A, b \in B\}$ is clearly a left (respectively right or two sided) ideal, called the sum of A and B. We can say every finite sum of left (respectively right or two sided) ideal of a Γ-ring is also a left (respectively right or two sided) ideal.

Matrix Gamma Ring: Let M be a Γ-ring and let $M_{m,n}$ and $\Gamma_{n,m}$ denote, respectively, the sets of $m \times n$ matrices with entries from M and set of $n \times m$ matrices with entries from Γ, then $M_{m,n}$ is a $\Gamma_{n,m}$ ring and multiplication defined by

$$(a_{ij})(\gamma_{ij})(b_{ij}) = (c_{ij}), \text{ where } c_{ij} = \sum_p \sum_q a_{ip}\gamma_{pq}b_{qj}.$$ If m = n, then M_n is a Γ_n-ring.

Γ-ring with minimum condition: A Γ-ring M with identity element 1 is called a Γ-ring with minimum condition if the ideals of M satisfy the descending chain condition or equivalently if in every non empty set of left ideals of M, there exists a left ideal which does not properly contain any other ideal in the set.

Gamma Homomorphism: Let M and N be two Γ-rings. Let φ be a map from M to N. Then φ is a Γ-homomorphism if and only if $\varphi(x + y) = \varphi(x) + \varphi(y)$ and $\varphi(x\gamma y) = \varphi(x)\gamma\varphi(y)$ for all $x, y\in M$ and all $\gamma\in\Gamma$. If φ is one-one and onto, then φ is Γ-isomorphism and it is denoted by $M \cong N$.

Division gamma ring: Let M be a Γ-ring. Then M is called a division Γ-ring if it has an identity element and its only non zero ideal is itself.

Minimal left (right) ideal of a Γ-ring: Let M be a Γ-ring. A left (right) ideal A of M is called a minimal left (right) ideal if

i) $A \neq 0$

ii) whenever $A \supseteq J \supseteq 0$, J is a left (right) ideal of M, then either J = A or J = 0.

It is clear that if a Γ-ring $M \neq 0$ satisfies the minimum condition on left(right) ideals, then M has a minimal left (right) ideal.

Zorn's lemma: Let A be a nonempty partially ordered set in which every totally ordered subset has an upper bound in A. Then A contains at least one maximal element.

ΓM-module: Let M be a Γ-ring and let (P, +) be an abelian group. Then P is called a left ΓM-module if there exists a Γ-mapping (Γ-composition) from $M\times\Gamma\times P$ to P sending (m, α, p) to mαp such that

i) $(m_1 + m_2)\alpha p = m_1\alpha p + m_2\alpha p$

ii) $m\alpha(p_1 + p_2) = m\alpha p_1 + m\alpha p_2$

iii) $(m_1\alpha m_2)\beta p = m_1\alpha(m_2\beta p)$,

for all $p, p_1, p_2\in P$, $m, m_1, m_2\in M$, $\alpha, \beta\in\Gamma$.

If in addition, M has an identity 1 and $1\gamma p = p$ for all $p\in P$ and some $\gamma\in\Gamma$, then P is called a unital ΓM-module.

2. Γ-VECTOR SPACE

2.1. Definition. Let (V, +) be an abelian group. Let Δ be a division Γ-ring with identity 1 and let $\varphi: \Delta\times\Gamma\times V\to V$, where we denote $\varphi(\delta,\gamma,v)$ by δγv. Then V is called a **left Γ-vector space** over Δ, if for all $\delta_1, \delta_2\in\Delta$, $v_1, v_2\in V$ and $\beta, \gamma\in\Gamma$, the following hold:

i) $\delta_1\gamma(v_1 +v_2) = \delta_1\gamma v_1 + \delta_2\gamma v_2$

ii) $(\delta_1 + \delta_2)\gamma v_1 = \delta_1\gamma v_1 + \delta_2\gamma v_1$

iii) $(\delta_1\beta\delta_2)\gamma v_1 = \delta_1\beta(\delta_2\gamma v_1)$

iv) $1\gamma v_1 = v_1$ for some $\gamma\in\Gamma$.

We call the elements v of V vectors and the elements δ of Δ scalars. We also call δγv the scalar multiple of v by δ. Similarly, we can also define **right Γ-vector space** over **Δ.**

2.2. Example: A left (respectively right) Γ-module over a division Γ-ring Δ is a left (respectively right) Γ-vector space over Δ.

2.3. Definition. Let V be a left Γ-vector space over Δ. A non empty sub set U of V is called a **sub Γ-Space of V** if (i)(U, +) is a sub group of (V, +) (ii)$\delta\gamma u\in U$ for all $\delta\in\Delta$, $\gamma\in\Gamma$, $u\in U$.

It is clear that U is a sub Γ-space of V provided that U is closed with respect to the operations of addition in V and scalar multiplication of vectors by scalars.

2.4. Definition. Let V be a left Γ-vector space over a division Γ-ring Δ. Let $v_1, v_2, \ldots, v_n\in V$ and for $\gamma\in\Gamma$, the vector $v\in V$ can be written as $v = \delta_1\gamma v_1 + \delta_1\gamma v_2 + \ldots + \delta_n\gamma v_n$, $\delta_1, \delta_2, \ldots, \delta_n \in\Delta$ is called a **linear γ-combination** of the v_i's over Δ. If v is a linear γ-combination for some $\gamma\in\Gamma$, then v is called a **linear Γ-combination** of the v_i's over Δ.

2.5. Definition. Let V be a left Γ-vector space over a division Γ-ring Δ. For $\gamma\in\Gamma$, then the set of vectors $\{v_i \mid i\in\Lambda\}$ is called **linearly γ-independent over Δ** (or simply γ-independent) if for each finite sub set of vectors $v_{i_1}, v_{i_2}, \ldots, v_{i_n}$ of $\{v_i \mid i \in \Lambda\}$, $\delta_1\gamma v_{i_1} + \delta_2\gamma v_{i_2} + \ldots\ldots + \delta_n\gamma v_{i_n} = 0$ implies $\delta_1 = \delta_2 = \ldots\ldots = \delta_n = 0$. Otherwise, the set $\{v_i \mid i \in \Lambda\}$ is called **linearly γ-dependent** (or simply **γ-dependent**). If $\{v_i \mid i \in \Lambda\}$ is **γ-independent** for some $\gamma \in \Gamma$, then $\{v_i \mid i \in \Lambda\}$ is called **linearly Γ-independent**. Otherwise the set $\{v_i \mid i \in \Lambda\}$ is called **linearly Γ-dependent.**

2.6. Definition. Let V be a left Γ-vector space over a division Γ-ring Δ. Let G be a sub set of V. Let G = $\{v_i\}$. Then G is said to be a set of **generators** for V or G **spans** V, if any $v \in V$ is a linear Γ-combination of vectors in G.

2.7. Definition. Let V be a left Γ-vector space over a division Γ-ring Δ. A **basis** B for V is a subset of V such that

(i) B spans V and

(ii) B is Γ-independent.

As a consequence of these definitions, we obtain the following results:

2.8. Theorem. Let V be a left Γ-vector space over a division Γ-ring Δ and let B be a basis of V. Then if $v \in V$, $v \neq 0$, there exist unique vectors $v_{i_1}, v_{i_2}, \ldots, v_{i_m} \in B$ and unique non zero scalars $\delta_1, \delta_2, \ldots, \delta_m \in \Delta$ such that $v = \delta_1\gamma v_1 + \delta_2\gamma v_2 + \ldots + \delta_m\gamma v_m$ for unique $\gamma \in \Gamma$.

Proof: Suppose $v = \delta_1\gamma v_{i_1} + \delta_1\gamma v_{i_2} + \ldots + \delta_m\gamma v_{i_m} = k_1\gamma v_{j_1} + k_1\gamma v_{j_2} + \ldots + k_n\gamma v_{j_n}$. By filling in each expression with $0\gamma v_j$'s and $0\gamma v_i$'s respectively, we get

$$v = \delta_1\gamma v_{i_1} + \delta_2\gamma v_{i_2} + \ldots + \delta_m\gamma v_{i_m} + 0\gamma v_{j_1} + 0\gamma v_{j_2} + \ldots\ldots + 0\gamma v_{j_n}$$

$$= 0\gamma v_{i_1} + 0\gamma v_{i_2} + \ldots\ldots + 0\gamma v_{i_m} + k_1\gamma v_{j_1} + k_2\gamma v_{j_2} + \ldots\ldots + k_n\gamma v_{j_n}.$$

In each expression v is a linear γ-combination of the same vectors. Hence $v = \delta_1\gamma v_{i_1} + \delta_2\gamma v_{i_2} + \ldots + \delta_n\gamma v_{i_n} = k_1\gamma v_{i_1} + k_1\gamma v_{i_1} \ldots\ldots + k_n\gamma v_{i_n}$.

Then $(\delta_1\gamma v_{i_1} - k_1\gamma v_{i_2}) + (\delta_2\gamma v_{i_2} - k_2\gamma v_{i_2}) + \ldots + (\delta_n\gamma v_{i_n} - k_n\gamma v_{i_n}) = 0$

$\Rightarrow (\delta_1 - k_1)\gamma v_{i_1} + (\delta_2 - k_2)\gamma v_{i_2} + \ldots + (\delta_n - k_n)\gamma v_{i_n} = 0$.

Since B is a Γ-independent set, then $\delta_i - k_i = 0$; i = 1,2 , . . . , n. Therefore $\delta_i = k_i$; i = 1, 2, 3, . . . , n. Hence the theorem is proved.

2.9. Definition. Let V be a left Γ-vector space over a division Γ-ring Δ. A set H = $\{v_i \mid i \in \Lambda\}$of linearly Γ-independent vectors in V is called a **maximal set of linearly Γ-independent vectors** in V if whenever $H \subset D \subseteq V$ (and D has no repetitions), then D is a Γ-dependent set.

2.10. Definition. Let V be a left Γ-vector space over a division Γ-ring Δ. A set G (without repetitions) of generators of V is called a **minimal set of generators** if whenever $H \subset G$, then H is not a set of generators of V.

2.11. Definition. Let V be a left Γ-vector space over a division Γ-ring Δ. If V has a basis with n elements, then we say that V is **finite dimensional** of dimension n over Δ and we denote this by [V : Δ] = n. If V does not have a finite basis, then we say that V is **infinite dimensional** and write [V : Δ] = ∞. We note that if V = {0}, then [V : Δ] = 0, since empty set is a basis for {0}.

2.12. Theorem. Let V be a left Γ-vector space over a division Γ-ring Δ. Let $B \subseteq V$. Then the following three conditions are equivalent:

(i) B is a basis for V

(ii) B is a minimal set of generators for V

(iii) B is a maximal set of linearly Γ-independent vectors.

Proof: We will give a cyclic proof, that is, we will show that (i) implies (ii), (ii) implies (iii) and (iii) implies (i). Without loss of generality we may assume that $V \neq 0$. For if V = 0, then B is a empty set satisfies (i), (ii) and (iii).

(i) implies (ii). Since B is a basis of V, then clearly B is a set of generators. Now let $H \subset B$ and suppose $b_i \in B$, but $b_i \notin H$. We must show that H is not a set of generators for V. If it were, then there would exist scalars $\delta_1, \delta_2, \ldots, \delta_j$ such that $b_i = \delta_1\gamma\, b_1 + \delta_2\gamma\, b_2 + \ldots + \delta_j\gamma\, b_j$, where $b_1, b_2, \ldots, b_j \in B$, $b_i \neq b_k$, k = 1, 2, . . ., j and for some $\gamma\in\Gamma$. Thus b_i is represented as a linear Γ-combination of vectors of B in two different ways ($b_i = 1\gamma b_i$ and $b_i = \delta_1\gamma b_1 + \delta_1\gamma b_2 + \ldots + \delta_j\gamma b_j$). By Theorem **2.8,** which contradicts that B is a basis of V. Thus H does not generate V. Hence B is a minimal set of generators for V.

(ii) implies (iii). First we show that B is a set of Γ-independent vectors. Since $V\neq 0$, then it is clear that $0\notin B$ and that B is non empty . For if $0\in B$, we can delete 0 and still have a set of generators. Now if B is not a Γ-independent set, then there exist vectors $b_1, b_2, \ldots, b_k$ in B and scalars $\delta_1, \delta_2, \ldots, \delta_k$ such that $b_i = \delta_2\gamma b_2 + \delta_3\gamma b_3 + \ldots + \delta_k\gamma b_k$ for some $\gamma\in\Gamma$. But then clearly we can delete b_1 from B and still have a set of generators, which contradicts the minimality of B. Thus B is a Γ-independent set.

Now we must show that B is a maximal independent set. Let $B\subset H$ and let $h\in H$, $h\notin B$. Since B is a set of generators, then $h = \delta_1\gamma b_1 + \delta_2\gamma b_2 + \ldots + \delta_k\gamma b_k$ for some $\delta_1, \delta_2, \ldots, \delta_k\in\Delta$, $b_1, b_2, \ldots, b_k\in B$ and some $\gamma\in\Gamma$. Hence H is a Γ-dependent set of vectors. Thus B is a maximal set of linearly Γ-independent vectors.

(iii) implies (i). Since B is a Γ-independent set, then we only need to show that B generates V. Let $v\in V$. If we cannot write $v = \delta_1\gamma b_1 + \delta_2\gamma b_2 + \ldots + \delta_k\gamma b_k$ for some choice $\delta_1, \delta_2, \ldots, \delta_k\in\Delta$, $b_1, b_2, \ldots, b_k\in B$ and unique $\gamma\in\Gamma$, then the set $B\cup\{v\}$ is a Γ-independent set of vectors, which contradicts the maximality of B. Thus v can be written as a linear γ-combination of elements of B for unique $\gamma\in\Gamma$. Hence B is a basis of V. Thus the theorem is proved.

2.13. Theorem. Let V be a left Γ-vector space over a division Γ-ring Δ. Let $\{v_1, v_2, \ldots, v_k\}$ be a set of linearly Γ-independent vectors. Let $u_1, u_2, \ldots, u_{k+1}$ be k+1 vectors, each of which is a linear Γ-combination of v_i's. Then $\{u_1, u_2, \ldots, u_{k+1}\}$ is a linearly Γ-dependent set of vectors.

Proof: The proof is by induction on k.

Suppose k =1.Then $u_1 = a_1\gamma v_1$ and $u_2 = a_2\gamma v_1$ for some $\gamma\in\Gamma$. If either $u_1 = 0$ or $u_2 = 0$, then $v_1 = 0$, since a_1 and a_2 are not zero. Then the result is trivial. If $u_1 \neq 0$, then

$$a_1^{-1}\gamma u_1 = a_1^{-1}\gamma(a_1\gamma v_1) = (a_1^{-1}\gamma a_1)\gamma v_1 = 1\gamma v_1 = v_1.$$

Again if $u_2\neq 0$, then $a_2^{-1}\gamma u_2 = a_2^{-1}\gamma(a_2\gamma v_1) = (a_2^{-1}\gamma a_2)\gamma v_1 = 1\gamma v_1 = v_1$. Therefore $a_1^{-1}\gamma u_1 = a_2^{-1}\gamma u_2$

$\Rightarrow a_1\gamma(a_1^{-1}\gamma u_1) = a_1\gamma(a_2^{-1}\gamma u_2) \Rightarrow a_1\gamma a_1^{-1}\gamma u_1 = a_1\gamma a_2^{-1}\gamma u_2 \Rightarrow 1\gamma u_1 = a_1\gamma a_2^{-1}\gamma u_2 \Rightarrow u_1 = a_1\gamma a_2^{-1}\gamma u_2$. Thus the result holds.

Now suppose the result holds for all integer k, k < n.

Then we let,
$$u_1 = a_{11}\gamma v_1 + a_{12}\gamma v_2 + \ldots + a_{1n}\gamma v_n \quad \ldots (1)$$
$$u_2 = a_{21}\gamma v_1 + a_{22}\gamma v_2 + \ldots + a_{2n}\gamma v_n \quad \ldots (2)$$
. . . .

. . . .
$$u_{n+1} = a_{n+1,1}\gamma v_1 + a_{n+1,2}\gamma v_2 + \ldots + a_{n+1,n}\gamma v_n \quad \ldots (n + 1).$$

Since we can assume that no $u_i = 0$ and we may assume $a_{1n}\neq 0$. Then in $u_1 = a_{11}\gamma v_1 + a_{12}\gamma v_2 + \ldots + a_{1n}\gamma v_n$, we can solve for v_n in terms of $u_1, v_1, v_2, \ldots, v_{n-1}$. Therefore we get

$$a_{1n}\gamma v_n = u_1 - a_{11}\gamma v_1 - a_{12}\gamma v_2 - \ldots - a_{1,n-1}\gamma v_{n-1}$$
$$\Rightarrow a_{1n}^{-1}\gamma(a_{1n}\gamma v_n) = a_{1n}^{-1}\gamma(u_1 - a_{11}\gamma v_1 - a_{12}\gamma v_2 - \ldots - a_{1,n-1}\gamma v_{n-1}).$$
$$\Rightarrow (a_{1n}^{-1}\gamma a_{1n})\gamma v_n = a_{1n}^{-1}\gamma u_1 - a_{1n}^{-1}\gamma a_{11}\gamma v_1 - a_{1n}^{-1}\gamma a_{12}\gamma v_2 - \ldots - a_{1n}^{-1}\gamma a_{1,n-1}\gamma v_{n-1}$$
$$\Rightarrow 1\gamma v_n = a_{1n}^{-1}\gamma u_1 - a_{1n}^{-1}\gamma a_{11}\gamma v_1 - a_{1n}^{-1}\gamma a_{12}\gamma v_2 - \ldots - a_{1n}^{-1}\gamma a_{1,n-1}\gamma v_{n-1}$$
$$\Rightarrow v_n = a_{1n}^{-1}\gamma u_1 - a_{1n}^{-1}\gamma a_{11}\gamma v_1 - a_{1n}^{-1}\gamma a_{12}\gamma v_2 - \ldots - a_{1n}^{-1}\gamma a_{1,n-1}\gamma v_{n-1}.$$

Substituting this expression for v_n in $u_2 = a_{21}\gamma v_1 + a_{22}\gamma v_2 + \ldots + a_{2n}\gamma v_n$ we get

$$u_2 = a_{21}\gamma v_1 + a_{22}\gamma v_2 + \ldots + a_{2n}\gamma(a_{1n}^{-1}\gamma u_1 - a_{1n}^{-1}\gamma a_{11}\gamma v_1 - a_{1n}^{-1}\gamma a_{12}\gamma v_2 - \ldots - a_{1n}^{-1}\gamma a_{1,n-1}\gamma v_{n-1})$$
$$\Rightarrow u_2 = a_{2n}\gamma v_1 a_{1n}^{-1}\gamma u_1 + (a_{21} - a_{2n}\gamma a_{1n}^{-1}\gamma a_{11})\gamma v_1 + (a_{22} - a_{2n}\gamma a_{1n}^{-1}\gamma a_{12})\gamma v_2 + \ldots + (a_{2,n-1} - a_{2n}\gamma a_{1n}^{-1}\gamma a_{1,n-1})\gamma v_{n-1}$$

Therefore u_2 can be written in terms of $v_1, v_2, \ldots, v_{n-1}, u_1$. Similarly $u_3, u_4, \ldots, u_{n+1}$ can be written in the terms of $v_1, v_2, \ldots, v_{n-1}, u_1$. Then these substitutions we have

$u_2 - a_{2n}\gamma a_{1n}^{-1}\gamma u_1, u_3 - a_{3n}\gamma a_{1n}^{-1}\gamma u_1, \ldots, u_{n+1} - a_{n+1}\gamma a_{1n}^{-1}\gamma u_1$ written as linear Γ-combination of $v_1, v_2, \ldots, v_{n-1}$. Since $\{v_1, v_2, \ldots, v_{n-1}\}$ is a Γ-independent set, then by induction we have that the n vectors $u_i - a_{in}\gamma a_{1n}^{-1}\gamma u_1$, $i = 2, 3, \ldots, (n+1)$ are Γ-dependent. Thus there exist scalars $b_2, b_3, \ldots, b_{n+1}$ not all zero such that

$$b_2\gamma(u_2 - a_{2n}\gamma a_{1n}^{-1}\gamma u_1) + b_3\gamma(u_3 - a_{3n}\gamma a_{1n}^{-1}\gamma u_1) + \ldots + b_{n+1}\gamma(u_{n+1} - a_{n+1,n}\gamma a_{1n}^{-1}\gamma u_1 = 0.$$
$$\Rightarrow b_2\gamma u_2 + b_3\gamma u_3 + \ldots + b_{n+1}\gamma u_{n+1} + (-b_2\gamma a_{2n}\gamma a_{1n}^{-1} - \ldots - b_{n+1}\gamma a_{n+1,n}\gamma a_{1n}^{-1})\gamma u_1 = 0.$$

Hence $\{u_1, u_2, \ldots, u_{n+1}\}$ is a Γ-dependent set of vectors. Thus the theorem is proved.

2.14. Theorem. Let $V \neq 0$ be a left Γ-vector space over a division Γ-ring Δ. Then V has a basis.

Proof: Recall the definition of a Γ-independent set of vectors. Let F be the family of all Γ-independent subsets of V. Clearly F is nonempty, for, if $v \neq 0$, then $\{v\}$ is a Γ-independent set. We partially order F by set inclusion, that is, $B_1 \leq B_2$ if and only if $B_1 \subseteq B_2$. Now let C be a chain in F. Let $B = \bigcup_{B_i \in C} B_i$. Then B is also a Γ-independent set. For, if it is not, we can find vectors $v_1, v_2, \ldots, v_k$ in B that are Γ-dependent. But there must be some B_i that contains $v_1, v_2, \ldots, v_k$, since B is just a union of a chain of sets. The Γ-dependence relation among $v_1, v_2, \ldots, v_k$ in B contradicts their Γ-independence in B_i. Thus B is Γ-independent and hence B is an upper bound in F for C.

By Zorn's Lemma, F has a maximal element H(say). We claim H is a basis for V. To see this, first observe H is a Γ-independent set of vectors. Next, let $v \in V$. If v is not a linear Γ- combination of vectors of H, then $H \cup \{v\}$ is a Γ-independent set, but this contradicts the maximality of H in F. Thus H is a maximal Γ-independent set of vectors. By Theorem **2.12**, H is a basis of V. Hence the theorem is proved.

2.15 Definition. Let V and U be a left Γ-vector spaces over a division Γ-ring Δ. Let $T: V \to U$ satisfy

(i) $T(v_1+v_2) = T(v_1) + T(v_2)$ for all $v_1, v_2 \in V$

(ii) $T(\delta\gamma v) = \delta\gamma T(v)$ for all $\delta \in \Delta, \gamma \in \Gamma, v \in V$.

We call T a **linear Γ-transformation** from V to U and we denote the set of all linear Γ-transformations from V to U by $Hom_\Delta(V, U)$. $Hom_\Delta(V, U)$ is an additive group.

For all $T, S \in Hom_\Delta(V,U)$, $T + S$ and $T\gamma S$ are respectively defined by

$(T + S)(x) = T(x) + S(x)$ and

$(T\gamma S)(x) = T(\gamma S(x))$ for all $x \in V$ and $\gamma \in \Gamma$.

2.16. Theorem(Main Result-1). Let V and U be the left Γ-vector spaces over a division Γ-ring Δ. Then $Hom_\Delta(V, U)$ is a Γ-ring.

Proof: (1) Let $T_1, T_2, T_3 \in Hom_\Delta(V, U)$.

Then $((T_1 + T_2)\gamma T_3)(x) = (T_1 + T_2)(\gamma T_3(x))$ for all $x \in V, \gamma \in \Gamma$.

$= T_1(\gamma T_3(x)) + T_2(\gamma T_3(x))$

$= (T_1\gamma T_3)(x) + (T_2\gamma T_3)(x)$

$= (T_1\gamma T_3 + T_2\gamma T_3)(x)$.

$\Rightarrow (T_1 + T_2)\gamma T_3 = T_1\gamma T_3 + T_2\gamma T_3$.

Let $T_1, T_2 \in Hom_\Delta(V, U)$.

Then $(T_1(\alpha + \beta)T_2)(x) = (T_1(\alpha + \beta)T_2)(x)$ for all $\alpha, \beta \in \Gamma$ and all $x \in V$

$= T_1((\alpha + \beta)(T_2(x)))$

$= T_1((\alpha T_2(x)) + (\beta T_2(x)))$

$= T_1(\alpha T_2(x)) + T_1(\beta T_2(x))$

$= (T_1\alpha T_2)(x) + (T_1\beta T_2)(x)$

$= (T_1\alpha T_2 + T_1\beta T_2)(x)$

$\Rightarrow (T_1(\alpha + \beta)T_2) = T_1\alpha T_2 + T_1\beta T_2$.

Let $T_1, T_2, T_3 \in Hom_\Delta(V, U)$.

Then $(T_1\alpha(T_2 + T_3)(x) = T_1(\alpha(T_2 + T_3)(x))$

$= T_1(\alpha(T_2(x)+T_3(x)))$

$= T_1(\alpha T_2(x) + \alpha T_3(x))$

$= T_1(\alpha T_2(x))+T_1(\alpha T_3(x))$

$= (T_1\alpha T_2)(x)+ (T_1\alpha T_3)(x) = (T_1\alpha T_2 + T_1\alpha T_3)(x)$

$\Rightarrow T_1\alpha(T_1 + T_3) = T_1\alpha T_2 + T_1\alpha T_3$ for all $\alpha \in \Gamma$ and $x \in V$.

(ii) Let $T_1, T_2, T_3 \in Hom_\Delta(V, U)$.

Then $((T_1\alpha T_2)\beta T_3)(x) = (T_1\alpha T_2)(\beta T_3(x))$

$= T_1(\alpha T_2(\beta T_3(x)))$

Again, $(T_1\alpha(T_2\beta T_3))(x) = T_1(\alpha(T_2\beta T_3)(x))$

$= T_1(\alpha T_2(\beta T_3)(x)))$

Thus $((T_1\alpha T_2)\beta T_3)(x) = T_1\alpha(T_2\beta T_3))(x)$

$\Rightarrow (T_1\alpha T_2)\beta T_3 = T_1\alpha(T_2\beta T_3)$.

Thus $Hom_\Delta(V, U)$ satisfies all the conditions of a Γ-ring. Hence $Hom_\Delta(V, U)$ is a Γ-ring. Thus the theorem is proved.

2.17 Theorem (Main Result-2): Let V and U be the left Γ-vector spaces with finite dimension n and m respectively over a division Γ-ring Δ. Then there is a 1-1 correspondence between the set $Hom_\Delta(V, U)$ and the set of all $n\times m$ matrices $\Delta_{n,m}$.

Proof: Let $E = \{e_i\}$ be a basis of V and let $T\in Hom_\Delta(V, U)$. Then for any $v\in V$, $T(v)$ is completely determined, if we know $T(e_i)$ for all $e_i\in E$. For if $v\in V$, $v \neq 0$, then there exist unique non zero scalars $\delta_1, \delta_2, \ldots, \delta_k$ in Δ and unique vectors $e_{i_1}, e_{i_2}, \ldots, e_{i_k}$ in E such that

$$v = \delta_1\gamma e_{i_1} + \delta_2\gamma e_{i_2} + \ldots + \delta_k\gamma e_{i_k} \text{ for unique } \gamma\in\Gamma.$$

Then $T(v) = T(\delta_1\gamma e_{i_1} + \delta_2\gamma e_{i_2} + \ldots + \delta_k\gamma e_{i_k})$

$= T(\delta_1\gamma e_{i_1}) + T(\delta_2\gamma e_{i_2}) + \ldots + T(\delta_k\gamma e_{i_k})$. by (i) of definition **2.15**

$= \delta_1\gamma T(e_{i_1}) + \delta_2\gamma T(e_{i_2}) + \ldots + \delta_k\gamma T(e_{i_k})$, by (ii) of definition **2.15.**

Moreover, it is possible to define a linear Γ-transformation T′ from V to U simply by defining the action of T′ on each of the e_i's and extending this definition according to (i) and (ii) of **3.15**; i.e., for each e_i in E, let $T'(e_i)$ be any vector of U. Once we have defined $T'(e_i)$, now we define for $v\in V$,

$$T'(v) = T'(\delta_1\gamma e_{i_1} + \delta_2\gamma e_{i_2} + \ldots + \delta_k\gamma e_{i_k})$$
$$= T'(\delta_1\gamma e_{i_1}) + T'(\delta_2\gamma e_{i_2}) + \ldots + T'(\delta_k\gamma e_{i_k})$$
$$= \delta_1\gamma T'(e_{i_1}) + \delta_2\gamma T'(e_{i_2}) + \ldots + \delta_k\gamma T'(e_{i_k}).$$

It is easy to verify that T′, defined in this manner, is indeed a linear Γ-transformation.

We now restrict our attention to the case where V and U are both finite dimensional. Thus let $\{e_1, e_2, \ldots, e_n\}$ be a basis of V and $\{f_1, f_2, \ldots, f_m\}$ be a basis of U, and let $T\in Hom_\Delta(V, U)$. Then for each i, $T(e_i)$ has a unique representation

$$T(e_i) = \delta_{i1}\gamma_{11}f_1 + \delta_{i2}\gamma_{22}f_2 + \ldots + \delta_{im}\gamma_{mm}f_m,$$

which we shall denote by $\sum_{j=1}^{m}\delta_{ij}\gamma_{jj}f_j$, where for unique $\gamma_{jj}\in\Gamma$. Given the scalars δ_{ij}; where i = 1, 2, ..., n and j = 1, 2, 3, ..., m, we associate the following rectangular array with T:

$$\begin{pmatrix} \delta_{11} & \delta_{12} & \ldots\ldots & \delta_{1m} \\ \delta_{21} & \delta_{22} & \ldots\ldots & \delta_{2m} \\ ' & ' & & ' \\ ' & ' & & ' \\ ' & ' & & ' \\ \delta_{n1} & \delta_{n2} & \ldots\ldots & \delta_{nm} \end{pmatrix}$$

This array is called an $n\times m$ matrix with coefficient in Δ, and we abbreviate it by $(\delta_{ij})_{n\times m}$ or by (δ_{ij}). Thus we see that given basis $\{e_1, e_2, \ldots, e_n\}$ for V, $\{f_1, f_2, \ldots, f_m\}$ for U, and linear Γ-transformation T, we obtain an $n\times m$ matrix. Conversely, if we are given an $n\times m$ matrix $(\delta_{ij})\in\Delta_{n,m}$, we define a linear Γ-transformation $T': V \rightarrow U$ in terms of the basis $\{e_1, e_2, \ldots, e_n\}$ and $\{f_1, f_2, \ldots, f_m\}$, as follows:

$T'(e_i) = \sum_{j=1}^{m}\delta_{ij}\gamma_{jj}f_j, i = 1, 2, \ldots, n$. Thus there exists a 1–1 correspondence between the set $Hom_\Delta(V, U)$ and $\Delta_{n,m}$, the set of all $n\times m$ matrices. Thus the lemma is proved.

Of our Special interest in the important situation where V = U and we study $Hom_\Delta(V, V)$.

Thus in particular case we have the following theorem:

2.18 Theorem (An Immediate Consequence of Theorem 3.17). Let V be an n dimensional left Γ-vector space over a division Γ-ring Δ. Then there is a 1-1correspondence between the set Δ_n of all n×n matrices over Δ and the set $Hom_{\Delta}(V, V)$.

2.19 Remark. The correspondence of the above Theorem is the desired Γ-ring isomorphism of $Hom_{\Delta}(V, V)$ and Δ_n. i.e., $Hom_{\Delta}(V, V) \cong \Delta_n$. Since $Hom_{\Delta}(V, V)$ is a Γ-ring, then Δ_n is also a Γ_n-ring.

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