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Original Article

Decomposition in Neotherian Gamma Rings

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ABSTRACT
Let M be a Γ-ring and Let R be a unital, finitely generated left ΓM-module. Suppose that N is a sub ΓM-module of R. Here we define primary radical of N and tertiary radical of N. Some characterizations of these two radicals are obtained. Finally we obtain the decomposition of Neotherian Gamma rings.

INTRODUCTION

The notion of a Γ-ring was first introduced by N. Nobusawa [7] and then Barnes [1] generalized the definition of Nobusawa's gamma rings. Now a day we consider as a Γ-ring which is given by Barnes [1].

May Mathematican workesd on Γ-rings and they obtained some remarkable results. I.N. Herstein [4] obtained some results on decomposition of Noetherian rings. J. A. Riley [8] worked on primary and tertiary decompositions of rings and he proved some fruitful results relating to this.

In this paper, we generalize some works of I.N. Herstein in Γ-rings. We obtain decomposition of Noetherian Γ-rings by means of sub ΓM-Modules.

2. PRELIMINARIES

2.1. Definitions.

Gamma Ring. Let M and Γ be two additive abelian groups. Suppose that there is a mapping from $M \times \Gamma \times M \rightarrow M$ (sending (x, α, y) into $x\alpha y$) such that

i) $(x + y)\alpha z = x\alpha z + y\alpha z$
$x(\alpha + \beta)z = x\alpha z + x\beta z$
$x\alpha(y + z) = x\alpha y + x\alpha z$

ii) $(x\alpha y)\beta z = x\alpha(y\beta z)$,

Where $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. Then M is called a Γ-ring. A Γ-ring M is called commutative if $a\gamma b = b\gamma a$ for all $a, b \in M$ and all $\gamma \in \Gamma$.

Identity element of a Γ-ring. Let M be a Γ-ring. M is called a Γ-ring with identity if there exists an element $e \in M$ such that $a\gamma e = e\gamma a = a$ for all $a \in M$ and some $\gamma \in \Gamma$.

We shall frequently denote e by 1 and when M is a Γ-ring with identity, we shall often write $1 \in M$. Note that not all Γ-rings have an identity. When a Γ-ring has an identity, then the identity is unique.

Ideal of Γ-rings. A subset A of the Γ-ring M is a left (right) ideal of M if A is an additive subgroup of M and $M\Gamma A = \{c\alpha a \mid c \in M, \alpha \in \Gamma, a \in A\}$($A\Gamma M$) is contained in A. If A is both a left and a right ideal of M, then we say that A is an ideal or two sided ideal of M.

Prime ideal. An ideal P of a Γ-ring M is prime if $P \neq M$ and for any ideals A and B of M, $A\Gamma B \subseteq P$, implies $A \subseteq P$ or $B \subseteq P$.

Nilpotent element. Let M be a Γ-ring. An element x of M is called nilpotent if for every $\gamma \in \Gamma$, there exists a positive integer $n = n(\gamma)$ such that $(x\gamma)^n x = (x\gamma x\gamma \ldots \gamma x\gamma)x = 0$.

Nil ideal. An ideal A of a Γ-ring M is a nil ideal if every element of A is nilpotent that is, for all $x \in A$ and every $\gamma \in \Gamma$, $(x\gamma)^n x = (x\gamma x\gamma .. \gamma x\gamma)x = 0$, where n depends on the particular element x of A.

Nilpotent ideal. An ideal A of a Γ-ring M is called nilpotent if $(A\Gamma)^n A = (A\Gamma A\Gamma \ldots \Gamma A\Gamma)A = 0$, where n is the least positive integer.

Γ-ring with minimum condition. A Γ-ring M with identity element 1 is called a Γ-ring with minimum condition if the left ideals of M satisfy the descending chain condition or equivalently if in every non-empty set of left ideals of M, there exists a left ideal which does not properly contain any other ideal in the set.

Radical of a Gamma Ring. Let M be a Γ-ring with minimum condition. The two-sided ideal which is the sum of all nilpotent left ideals of M is called the radical of M and is denoted by rad (M).

Gmma module. Let M be a Γ-ring and let (R, +) be an abelian group. Then R is called a left ΓM-module if there exists a Γ-mapping (Γ-composition) from M×Γ×R to R sending (m, α, r) to mαr such that

(i) $(m_1 + m_2)\alpha r = m_1\alpha r + m_2\alpha r$

(ii) $m\alpha(r_1 + r_2) = m\alpha r_1 + m\alpha r_2$

(iii) $(m_1\alpha m_2)\beta r = m_1\alpha(m_2\beta r)$,

for all $r, r_1, r_2 \in R$, $m, m_1, m_2 \in M$, $\alpha, \beta \in \Gamma$.

If in addition, M has an identity 1 and $1\gamma r = r$ for all $r \in R$ and some $\gamma \in \Gamma$, then R is called a unital ΓM-module. We define right ΓM-module analogously.

Sub- ΓM-module. Let M be a Γ-ring. Let R be a left ΓM-module. Let (Q, +) be a subgroup of (R, +). We call Q, a sub- ΓM-module of R if $m\gamma q \in Q$ for all $m \in M$, $q \in Q$ and $\gamma \in \Gamma$.

Finitely Generated ΓM-module. Let M be Γ-ring. A left ΓM-module R is called finitely generated if R can be generated by a finite set of elements, that is, R is finitely generated if and only if there exist finitely many elements $x_1, x_2, \ldots, x_n \in R$ such that for each $r \in R$ can be expressed as a linear Γ-combination $r = \sum_{i=1}^{n} m_i\gamma x_i$ of the x_i with coefficients $m_i \in M$ for some $\gamma \in \Gamma$.

In this paper, we consider all ΓM-modules R as unital, finitely generated left ΓM- modules.

3. Decomposition in Noetherian Gamma Rings

3. 1 Definition. A Γ-ring M is called left Noetherian if it satisfy the ascending chain condition on left ideals and will have unit elements. All ΓM- modules R will be unital, finitely generated left ΓM – modules.

3.2 Definition. An element $a \in M$ is an annihilating element for R if $a\Gamma N = 0$ for some sub - ΓM-module $N \neq 0$ of R. A left ideal I of M is an annihilating left ideal for R if $I\Gamma N = 0$ for some sub- Γ M-module $N \neq 0$ of R.

We denote by O (R) the ideal $\{a \in M | a\Gamma R = 0\}$ of M.

The next few definitions make sense for any Γ -ring and all or parts of the very early lemmas hold in this more general context, but all these will be of most interest in the Noetherian case.

3.3 Definition. If N is a sub- ΓM-module of R then the primary radical of N, written rad(N), in the intersection of all prime ideals of M which contain $O\left(R/N\right)$. N is a primary sub- ΓM-module of R if all annihilating elements for R/N are in rad (N).

A remark which holds for any Γ-ring M: if $x \in$ rad(N) then $(x\gamma)^t x \in O\left(R/N\right)$ for some t depending on x and $\gamma \in \Gamma$, This is true since the intersection all prime ideals of a Γ -ring is a nil ideal.

One further definition at this point:

3.4 Definition. The tertiary radical of N, N a sub- ΓM-module of R, written t-rad(N), is the set of all elements of M which are annihilating elements for all non-trivial sub- Γ M-modules of R/N.

One immediately sees that:

$$\text{t-rad}(N) = \{a \in M | \, for\ all\ p \notin N \text{ there exists } q \in R, q \notin N\ with\ \ a\Gamma M\Gamma q \subset N\}.$$

3.5 Lemma. Let N be a sub- ΓM-module of R; if $a_1, a_2, \ldots, a_n \in$ t-rad(N) then, given p∈R, $p \notin N$ there is a $q \in M\Gamma p,\ q \notin N$ such that $a_i \Gamma M \Gamma q \subset N$ for i = 1, 2,...................,n.

Proof. We go by induction on n. If n=1 this is merely the definition of t-rad(N). Suppose then that we have found a $r \in M\Gamma p, r \notin N$ such that $a_i \Gamma M \Gamma r \subset N$ for i = 1, 2, ..., n-1. Since $a_n \in$ t-rad(N) there is a $q \in M\Gamma r \subset M\Gamma p\gamma q \notin N$ such that $a_n \Gamma M \Gamma q \subset N$, $\gamma \in \Gamma$. However for 1<n, $a_i \Gamma M \Gamma q \subset a_i \Gamma M \Gamma r \subset N$, thereby the lemma is proved.

3.6 Corollary. t-rad(N) is a two-sided ideal of M.

Proof. From the very definition of t-rad(N) we immediately have that $a\Gamma M \subset$ t-rad(N) and $M\Gamma a \subset$ t-rad(N) for all a∈t-rad(N). To finish we merely need that a, b∈t-rad(N) forces a-b∈t-rad(N); this is however clear from the lemma 3.5.

3.7 Corollary. Let I = t-rad (N); given p∉N there is a $q \in M\Gamma p$ such that $q \notin N, I\Gamma q \subset N$. **Proof.** Since M is left Noetherian and I is an ideal of M,

I=M$\Gamma a_1 + M\Gamma a_2 + \ldots\ldots\ldots + M\Gamma a_n$ for appropriate a_i∈I. Pick by the lemma3.5, q∈$M\Gamma p,\ q \notin N$ such that $a_i \gamma$ q∈N for i = 1, 2, ...,n, $\gamma \in \Gamma$; then $I\Gamma q \subset N$.

3.8 Corollary. Let I = t-rad(N) and let $N^t = \{p \in R | I\Gamma p \subset N\}$. Then $N^t \supset N$ and $N^t \neq N$.

Proof. Since I is a 2-sided ideal of M, N^t is a sub-ΓM −module of R. Clearly $N^t \supset$ N. By Corallary 3.7 we can find $q \notin N$ such that $I\Gamma q \subset N$; since $q \in N^t$, $q \notin N$ this finishes the proof.

In fact, we have proved a good deal more about the nature of N^t for we know that in each $M\Gamma p,\ p \notin N$ there are elements of N^t. We formulize this lemma 3.9 but first a definition.

As usual and as used before in these notes, R is said to be an essential extension of A modulo B, B⊂A sub-Γ M-modules of R, if whenever C is a sub-Γ M-module of R such that $C \cap A \subset B$ then $C \subset B$. That is R/B the Γ M-module A/B meets all non-zero sub-ΓM -modules non-trivially.

3.9 Lemma. R is an essential extension of N^t modulo N.

Proof. This is immediate from Corollary 3.7.

Before proceeding, we examine the relation of rad (N) to t-rad (N).

3.10 Lemma. rad(N) ⊂ t-rad(N).

Proof. As we remarked earliar for all $x \in rad(N), (x\gamma)^{n(x)} x \in t-rad(N)$. Since M is Noetherian by a result of Levitzki (rad(N) Γ)^k(rad(N) ⊂t-rad(N). From the definition of t-rad(N) this implies that rad (N) ⊂t-rad(N).

In the commutative case we can now easily establish equality for these two radicals.

3.11 Theorem. If M is a commutative Noetherian Γ-ring and if N is a sub-ΓM-module of R then rad(N) = t-rad(N).

Proof. Lemma 3.10 already tells us rad(N) ⊂t-rad(N). Since M is Noetherian and R is a finitely generated unital ΓM -module, R has the ascending chain condition on sub-ΓM-modules. Let α∈t-rad(N) and let $J_n = \{p \in R | M\Gamma(a\gamma)^n a\gamma p \subset N,\ \gamma \in \Gamma\}$. Then J_n form an ascending chain of sub-ΓM -modules of R hence for some integer n, J_n= J_{n+1}. If (aγ)^{n}a$\Gamma R \not\subset N, \gamma \in \Gamma$ we can find p∈R with $(a\gamma)^n a\gamma p \notin N$; since $a \in$ t-rad(N) there is an m∈M with mγ(aγ)^n $a\gamma p \notin N$ but for which a$\Gamma M \Gamma m\gamma(a\gamma)^n a\gamma p \subset N$. Since M is commutative this yields $M\Gamma(a\gamma)^{n+1} a\gamma m\gamma p \subset N$, that is, mγ $p \in J_{n+1} = J_n$ hence (aγ)^{n}aγmγ $p \in N$, contrary to (aγ)^n aγmγ $p \notin N$.Thus a∈t-rad(N) implies

$(a\gamma)^k a \in$ rad(N). Since rad(N) is the intersection of prime ideals, this forces $a \in$ rad(N), that is, t-rad(N) $\subset$ rad(N). This proves the theorem.

3.12 Definition. N is a tertiary sub-ΓM-module of R if the annihilating elements for R/N are all in t-rad(N) .

A quick verification reveals that N is a tertiary sub-Γ M-module of R if and only if

t-rad(N) = $\{a \in M \mid there\, is\, a \;\; q \notin N \; with \; a\Gamma M\Gamma q \subset N\}$.

3.13 Lemma. Let N be a tertiary sub-ΓM-module of R; then P = t-rad(N) is a prime ideal of M.

Proof. By Lemma 3.9, since $N^t \neq N$, there exists a $q \in R$, $q \notin N$ such that for all $a \in P$, $a\Gamma M\Gamma q \subset N$. Since N is tertiary, P consists of all elements $a \in M$ such that $a\Gamma M\Gamma q \subset N$.

Suppose $x,y \in M$ are such that $x\Gamma M\Gamma y \subset P$; then $x\Gamma M\Gamma y\Gamma M\Gamma q \subset N$. If $y \notin P$ then $y\Gamma M\Gamma q \not\subset N$ so by the tertiary nature of N we get $x \in P$. Therefore $x\Gamma M\Gamma y \subset P$ implies that $x \in P$ or $y \in P$, hence P is a prime ideal of M.

If N is a tertiary sub-ΓM-module of R and the prime ideal P of M is t-rad(N) then we say that N is P-tertiary and that P is the prime ideal of N.

A sub-ΓM-module N of R is called irreducible if is not the intersection of two strickly larger sub-Γ M-modules.

3.14 Lemma. If N is an irreducible sub-ΓM-module of R, then it is tertiary,

Proof. If N is not tertiary there exist $a \notin$ t-rad(N), $p \notin N$ such that $a\Gamma M\Gamma p \subset N$. Since $a \notin$ t-rad(N) there is a $r \notin N$ with $a\Gamma M\Gamma r^t \subset N$ where $r^t \in M\Gamma r$ implies $r^t \in N$.

Let $N^* = (N+M\Gamma p) \cap (N+M\Gamma r)$; clearly $N^* \supset N$. For $q \in N^*$, $q = p^t + m\gamma q$

$= p^{tt} + s\gamma r$ with $p^t, p^{tt} \in N$, $\gamma \in \Gamma$. Thus $s\gamma r = (p^t - p^{tt}) + m\gamma p$, so

$a\Gamma M\Gamma s\gamma r \subset a\Gamma M\Gamma p \subset N$. Since $r^t = s\gamma r \in M\Gamma r$ satisfies $a\Gamma M\Gamma r^t \subset N$, we have that $r^t \in N$, hence $q \in N$.

Thus $N^* \subset N$. We have exhibited N as an intersection of larger sub-Γ M-modules. This proves the lemma.

3.15 Definition. The decomposition $N = N_1 \cap$ ------ $\cap N_m$ of N by the N_i is irredundant if no N_i can be omitted.

3.16 Lemma. If $N = N_1 \cap$ ------ $\cap N_m$ is an irredundant decomposition of N by P_i-tertiary sub-ΓM-modules N_i , i = 1, 2,-----, m then t-rad(N) = $P_1 \cap$ ---- $\cap P_m$.

Proof. Let $a \in P_1 \cap$ ------ $\cap P_m$ and let $p \in R$, $p \notin N$. Since $p \notin N$, p is not in some N_i say $p \notin N_1$. From the fact that $a \in$ t-rad(N_1) there is a $p_1 \in M\Gamma p$, $p_1 \notin N_1$ such that $a\Gamma M\Gamma p_1 \subset N_1$. If $p_1 \in N_2$ then $a\Gamma M\Gamma p_1 \subset N_1 \cap N_2$; if $p_1 \notin N_2$ there is an element $p_2 \in M\Gamma p_1 \subset M\Gamma p$, $p_2 \notin N_2$ such that $a\Gamma M\Gamma p_2 \subset N_2$. Since $a\Gamma M\Gamma p_2 \subset a\Gamma M\Gamma p_1 \subset N_1$ we have $a\Gamma M\Gamma p_2 \subset N_1 \cap N_2$. Continuing this way we get an element $q \in M\Gamma p$, $q \notin N$ such that $a\Gamma M\Gamma q \subset N_1 \cap$---$\cap N_m$= N. Thus by the definition of t-rad(N), $a \in$ t-rad(N) hence $P_1 \cap$------ $\cap P_m \subset$ t-rad(N).

Suppose now that $a \in$ t-rad(N) and that $p \in N_j$ for all $j \neq i$, $p \notin N_i$. Since N is the irredundant intersection of the N_i, $p \notin N$. Therefore there is a $q \in M\Gamma p$, $q \notin N$ such that $a\Gamma M\Gamma q \subset N \subset N_j$. Since $q \in N_j$ for $j \neq i$ and $q \notin N$ we must have that $q \notin N_i$. Since N_i is P_i -tertiary we conclude that $a \in P_i$. But then $a \in \bigcap_{i=1}^{m} P_i$, whence t-rad(N)$\subset P_1 \cap$----$\cap P_m$. Hence t-rad(N)=$P_1 \cap$-----$\cap P_m$.

3.17 Lemma. If N_1 and N_2 are P-tertiary sub-ΓM-modules of R then so is $N_1 \cap N_2$.

Proof . Let $N = N_1 \cap N_2$; by Lemma 3.13, t-rad(N) = P. To finish the proof we merely must show that N is tertiary. Let $a \in M$, $q \in R$, $q \notin N$ such that $a\Gamma M\Gamma q \subset N$. Since $q \notin N$, $q \notin N_1$ or $q \notin N_2$. If $q \notin N_1$, since N_1 is tertiary, $a \in$ t-rad(N_1) = P ; similarly if $q \notin N_2$, $a \in P$. Thus $a\Gamma M\Gamma q \subset N$, $q \notin N$ implies $a \in P$ = t-rad(N). Hence N is tertiary.

3.18 Definition. A decomposition $N = N_1 \cap \text{----} \cap N_m$ of a sub-ΓM-module N by P_i -tertiary sub-ΓM-modules N_i is called reduced if it is irredundant and P_i are distinct.
We now can prove the analog of the primary decomposition for commutative Noetherian Γ-rings.

3.19 Theorem . Every sub-Γ M-module N of R has a reduced decomposition $N= N_1\cap\text{------}\cap N_m$ where N_i is P_i-tertiary.
Proof. That every sub-ΓM-module has an intersection of a finite number of irreducible ones is easy, just as in the commutative case. Merely consider the set of sub-ΓM-modules which have no such representation; if this set is non-empty it has a maximal element. However, this maximal element is therefore irreducible, giving a contradiction. The rest follows from lemmas 3.13, 3.14 3.16 and 3.17.
We now want to establish the uniqueness of the associated primes.To this end we prove

3.20 Lemma. If N, U, V, U^t, V^t are sub-ΓM-modules of R such that U is P-tertiary, U^t is P^t-tertiary, $P\neq P^t$ and $N = U\cap V= U^t\cap V^t$ then $N = V\cap V^t$.
Proof. Let $p\in V\cap V^t$. Since $P\neq P^t$, there is an element in one and not in the other; say $a\in P$, $a\notin P^t$. If $p\in N$ there exists a $q\in M\Gamma p$, $q\notin N$ such that $a\Gamma M\Gamma q\subset N$. Thus $a\Gamma M\Gamma q\subset U^t\cap V^t$. Since $q\notin N$, $q\notin U^t$. Since U^t is P^t-tertiary and $a\Gamma M\Gamma q\subset U^t$ we conclude that $a\in P^t$, a contradiction. Thus $p\in N$. That is. $V\cap V^t\subset N$; since clearly $N\subset V$, $N\subset V^t$ we get $N\subset V\cap V^t$. This proves the lemma.
The lemma immediately yields

3.21 Theorem. If the sub-ΓM-module N of R has the two reduced decompositions $N=N_1\cap N_2\cap \text{-----}\cap N_m = N_1^t\cap \text{-----}\cap N_s^t$ then m = s and the set of prime ideals P_i of the N_i concides with the set of prime ideals $P_i^t$of the N_i^t .

Proof. We show $P_i^t = P_i$ for some i. Suppose not. Since $P_1^t = P_i$ and $N = N_1\cap \text{---} \cap N_m =N_1^t\cap \text{----}\cap N_s^t$, by Lemma 3.20, $N = N_2\cap\text{------}\cap N_m\cap N_2^t\cap\text{-----}\cap N_s^t$. Since $P_i^t \neq P_2$ we get $N= N_3\cap\text{------}\cap N_m\cap N_2^t \cap\text{-----}\cap N_s^t$. Continuing we arrive at $N = N_2^t\cap\text{----}\cap N_s^t$ contrary to the irredundancy of the representation $N = N_1^t \cap\text{------}\cap N_s^t$. Thus $P_i^t = P_i$. In the same way given j, $P_j^t = P_k$ for some k. This shows $s\leq m$. The argument is symmetric, $m\leq s$. Thus m = s and $\{P_i^t\} = \{P_i\}$.

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