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Original Article

On Degree of Approximation of Conjugate Series of Fourier Series by Product Means $(E,q)A$

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ABSTRACT

In this paper a theorem on degree of Approximation of a function $f \in Lip\ \alpha$ by product summability $(E,q)A$ of conjugate series of Fourier series associated with f.

KEYWORDS: Degree of Approximation, $Lip\ \alpha$ class of function, (E,q)- mean, A - mean, $(E,q)A$ -product mean, conjugate Fourier series, Lebesgue integral.
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INTRODUCTION

Let $\sum a_n$ be a given infinite series with the sequence of partial sums $\{s_n\}$. Let $A = (a_{mn})_{\infty\times\infty}$ be a matrix .Then the sequence –to-sequence transformation

(1.1) $$t_n = \sum_{\upsilon=0}^{n} a_{m\nu}\, s_\upsilon \ ,n = 1,2,\cdots$$

defines the sequence $\{t_n\}$ of the A -mean of the sequence $\{s_n\}$. If

(1.2) $$t_n \to s \quad ,\text{as} \quad n \to \infty \ ,$$

then the series $\sum a_n$ is said to be A summable to s .

The conditions for regularity of A -summability are easily seen to be

(i) $\sup_m \sum_{n=0}^{\infty} |a_{mn}| < H$ where H is an absolute constant.

(ii) $\lim_{m\to\infty} a_{mn} = 0$

(iii) $\lim_{m\to\infty} \sum_{n=0}^{\infty} a_{mn} = 1$

The sequence –to-sequence transformation, [1]

(1.3) $$T_n = \frac{1}{(1+q)^n} \sum_{\upsilon=0}^{n} \binom{n}{\upsilon} q^{n-\upsilon} s_\upsilon \quad .$$

defines the sequence $\{T_n\}$ of the A mean of the sequence $\{s_n\}$.

If

(1.4) $T_n \to s$, as $n \to \infty$,

then the series $\sum a_n$ is said to be (E,q) sum able to s.

Clearly (E,q) method is regular [**1**]. Further, the (E,q) transform of the A transform of $\{s_n\}$ is defined by

$$\tau_n = \frac{1}{(1+q)^n}\sum_{k=0}^{n}\binom{n}{k} q^{n-k}\, t_k$$

(1.5) $$= \frac{1}{(1+q)^n}\sum_{k=0}^{n}\binom{n}{k} q^{n-k}\left\{\sum_{\upsilon=0}^{k} a_{k\upsilon}s_\upsilon\right\}$$

If

(1.6) $\tau_n \to s$, as $n \to \infty$,

then $\sum a_n$ is said to be $(E,q)(N,p_n)$-summable to s.

Let $f(t)$ be a periodic function with period 2π, L-integrable over (-π,π), The Fourier series associated with f at any point x is defined by

(1.7) $$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \equiv \sum_{n=0}^{\infty} A_n(x)$$

and the conjugate series of the Fourier series (1.8) is

(1.8) $$\sum_{n=1}^{\infty}(a_n \cos nx - b_n \sin nx) \equiv \sum_{n=1}^{\infty} B_n(x)$$

Let $\overline{s_n}\ (f;x)$ be the n-th partial sum of (1.9).

The L_∞-norm of a function $f : R \to R$ is defined by

(1.9) $$\|f\|_\infty = \sup\{|f(x)| : x \in R\}$$

and the L_υ-norm is defined by

(1.10) $$\|f\|_\upsilon = \left(\int_0^{2\pi}|f(x)|^\upsilon\right)^{\frac{1}{\upsilon}} \quad , \upsilon \ge 1$$

The degree of approximation of a function $f : R \to R$ by a trigonometric polynomial $P_n(x)$ of degree n under norm $\|\ .\ \|_\infty$ is defined by [**3**].

(1.11) $$\|P_n - f\|_\infty = \sup\{|P_n(x) - f(x)| : x \in R\}$$

and the degree of approximation $E_n(f)$ of a function $f \in L_\upsilon$ is given by

(1.12) $$E_n(f) = \min_{P_n}\|P_n - f\|_\nu$$

This method of approximation is called trigonometric Fourier approximation.

A function $f \in Lip\alpha$ if

(1.13) $$|f(x+t) - f(x)| = O(|t|^\alpha) \ ,0 < \alpha \le 1$$

We use the following notation throughout this paper:

(1.14) $$\psi(t) = \frac{1}{2}\{f(x+t) - f(x-t)\},$$

and

(1.15) $$\overline{K_n}(t)=\frac{1}{\pi(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\sum_{\upsilon=0}^{k}a_{k\upsilon}\frac{\cos\frac{t}{2}-\cos\left(\upsilon+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}.$$

Further, the method $(E,q)A$ is assumed to be regular and this case is supposed through out the paper.

KNOWN THEOREM

Dealing with the degree of approximation by the product $(E,q)(C,1)$-mean of Fourier series, Nigam [**2**] proved the following theorem.

Theorem- 2.1:

If a function f ,2π - periodic ,belonging to class $Lip\alpha$, then its degree of approximation by $(E,q)(C,1)$ summability mean on its Fourier series $\sum_{n=0}^{\infty}A_n(t)$ is given by

$\left\|E_n^qC_n^1-f\right\|_\infty=O\left(\frac{1}{(n+1)^\alpha}\right)$,$0<\alpha<1$,where $E_n^qC_n^1$ represents the (E,q) transform of $(C,1)$ transform of $s_n(f;x)$.

MAIN THEOREM

In this paper, we have proved a theorem on degree of approximation by the product mean $(E,q)A$ of conjugate series of Fourier series of (1.8), we prove:

Theorem -3.1:

If f is a $2\pi-$ Periodic function of class $Lip\alpha$, then degree of approximation by the product $(E,q)A$ summability means on its conjugate series of Fourier series (1.8) is given by $\left\|\tau_n-f\right\|_\infty=O\left(\frac{1}{(n+1)^\alpha}\right)$,$0<\alpha<1$, where τ_n as defined in (1.5) .

LEMMAS

We require the following Lemmas to prove the theorem-3.1.

Lemma - 4.1:

$\left|\overline{K_n}(t)\right|=O(n)$,$0\le t\le\frac{1}{n+1}$, where $\overline{K_n}(t)$ is as defined in (1.15)

Proof of Lemma- 4.1:

For $0\le t\le\frac{1}{n+1}$, we have sinnt ≤ nsint , then

$$\left|\overline{K_n}(t)\right|=\frac{1}{\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\sum_{\upsilon=0}^{k}a_{k\upsilon}\frac{\cos\frac{t}{2}-\cos\left(\upsilon+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}\right|$$

$$\leq \frac{1}{\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k} q^{n-k}\left\{\sum_{\upsilon=0}^{k} a_{k\upsilon}\frac{\cos\frac{t}{2}-\cos\upsilon t.\cos\frac{t}{2}+\sin\upsilon t.\sin\frac{t}{2}}{\sin\frac{t}{2}}\right\}\right|$$

$$\leq \frac{1}{\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k} q^{n-k}\left\{\sum_{\upsilon=0}^{k} a_{k\upsilon}\left(\frac{\cos\frac{t}{2}\left(2\sin^2\upsilon\frac{t}{2}\right)}{\sin\frac{t}{2}}+\sin\upsilon t\right)\right\}\right|$$

$$\leq \frac{1}{\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k} q^{n-k}\left\{\sum_{\upsilon=0}^{k} a_{k\upsilon}\left(O\left(2\sin\upsilon\frac{t}{2}\,\sin\upsilon\frac{t}{2}\right)+\upsilon\sin t\right)\right\}\right|$$

$$\leq \frac{1}{\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k} q^{n-k}\left\{\sum_{\upsilon=0}^{k} a_{k\upsilon}\left(O(\upsilon)+O(\upsilon)\right)\right\}\right|$$

$$\leq \frac{1}{\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k} q^{n-k}O(k)\sum_{\upsilon=0}^{k} a_{k\upsilon}\right|$$

$$=\frac{H}{\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k} q^{n-k}O(k)\right|$$, by regularity condition

$= O(n)$.

This proves the lemma.

Lemma- 4.2:

$\left|\overline{K_n}(t)\right| = O\left(\frac{1}{t}\right), for\ \frac{1}{n+1}\leq t\leq\pi$, where $\overline{K_n}(t)$ is as defined in (1.15)

Proof of Lemma- 4.2:

For $\frac{1}{n+1}\leq t\leq\pi$, we have by Jordan's lemma, $\sin\left(\frac{t}{2}\right)\geq\frac{t}{\pi}$.

Then

$$\left|\overline{K_n}(t)\right| = \frac{1}{\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k} q^{n-k}\left\{\sum_{\upsilon=0}^{k} a_{k\upsilon}\frac{\cos\frac{t}{2}-\cos\left(\upsilon+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}\right|$$

$$=\frac{1}{\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k} q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k} p_{k-\upsilon}\frac{\cos\frac{t}{2}-\cos\upsilon\frac{t}{2}.\cos\frac{t}{2}+\sin\upsilon\frac{t}{2}.\sin\frac{t}{2}}{\sin\frac{t}{2}}\right\}\right|$$

$$\le \frac{1}{\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k} q^{n-k}\left\{\sum_{\upsilon=0}^{k}\frac{\pi}{2t}a_{k\upsilon}\cos\frac{t}{2}\left(2\sin^2\upsilon\frac{t}{2}\right)+\sin\upsilon\frac{t}{2}.\sin\frac{t}{2}\right\}\right|$$

$$\le \frac{\pi}{2\pi(1+q)^n t}\left|\sum_{k=0}^{n}\binom{n}{k} q^{n-k}\left\{\sum_{\upsilon=0}^{k}a_{k\upsilon}\right\}\right|$$

$$= \frac{1}{2(1+q)^n t}\left|\sum_{k=0}^{n}\binom{n}{k} q^{n-k}\left\{\sum_{\upsilon=0}^{k}a_{k\upsilon}\right\}\right|.$$

$$\le \frac{H}{2(1+q)^n t}\left|\sum_{k=0}^{n}\binom{n}{k} q^{n-k}\right|, \text{ by regularity condition}$$

$$= O\left(\frac{1}{t}\right).$$

This proves the lemma.

PROOF OF THEOREM -3.1

Using Riemann –Lebesgue theorem, we have for the n-th partial sum $\overline{s_n}(f;x)$ of the conjugate Fourier series (1.8) of,

$$\overline{s_n}(f;x)-f(x)=\frac{2}{\pi}\int_0^{\pi}\psi(t)\ \overline{K_n}\ dt,$$

following Titechmarch [**3**]the $A-$ transform of $\overline{s_n}(f;x)$ using (1.1) is given by

$$t_n - f(x) = \frac{2}{\pi}\int_0^{\pi}\psi(t)\sum_{k=0}^{n}a_{nk}\ \frac{\cos\frac{t}{2}-\sin\left(n+\frac{1}{2}\right)t}{2\sin\left(\frac{t}{2}\right)}\,dt,$$

denoting the $(E,q)A$ transform of $\overline{s_n}(f;x)$ by τ_n, we have

$$\left\|\tau_n - f\right\| = \frac{2}{\pi(1+q)^n}\int_0^{\pi}\psi(t)\sum_{k=0}^{n}\binom{n}{k} q^{n-k}\left\{\sum_{\upsilon=0}^{k}a_{k\upsilon}\ \frac{\cos\frac{t}{2}-\sin\left(\upsilon+\frac{1}{2}\right)t}{2\sin\left(\frac{t}{2}\right)}\right\}dt$$

$$= \int_0^{\pi}\psi(t)\ \overline{K_n}(t)\,dt$$

$$= \left\{\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi}\right\}\psi(t)\ \overline{K_n}(t)\ dt$$

(5.1) $\quad = I_1 + I_2$, *say*

Now

$$|I_1| = \frac{2}{\pi(1+q)^n}\left|\int_0^{1/n+1} \psi(t)\sum_{k=0}^{n}\binom{n}{k} q^{n-k}\left\{\sum_{\upsilon=0}^{k} a_{k\upsilon}\frac{\cos\frac{t}{2}-\cos\left(\upsilon+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}\right\}dt\right|$$

$$\leq O(n)\int_0^{\frac{1}{n+1}} |\psi(t)|\,dt \quad \text{, using Lemma 4.1}$$

$$= O(n)\int_0^{\frac{1}{n+1}} |\,t^{\alpha}|\,dt$$

$$= O(n)\left[\frac{t^{\alpha+1}}{\alpha+1}\right]_0^{\frac{1}{n+1}}$$

$$= O(n)\left[\frac{1}{(\alpha+1)(n+1)^{\alpha+1}}\right].$$

(5.2) $$= O\left[\frac{1}{(n+1)^{\alpha}}\right]$$

Next

$$|I_2| \leq \int_{\frac{1}{n+1}}^{\pi} |\psi(t)|\,|\overline{K_n}(t)|\,dt$$

$$= \int_{\frac{1}{n+1}}^{\pi} |\,\psi(t)\,|\,O\left(\frac{1}{t}\right)dt \quad \text{, using Lemma 4.2}$$

$$= \int_{\frac{1}{n+1}}^{\pi} |\,t^{\alpha}\,|\,O\left(\frac{1}{t}\right)dt$$

$$= \int_{\frac{1}{n+1}}^{\pi} t^{\alpha-1}\,dt$$

(5.5.3) $$= O\left(\frac{1}{(n+1)^{\alpha}}\right)$$

Then from (5.2) and (5.3) , we have

$$|\tau_n - f(x)| = O\left(\frac{1}{(n+1)^{\alpha}}\right) \text{, for } 0<\alpha<1.$$

Hence, $\left\|\tau_n - f(x)\right\|_\infty = \sup_{-\pi<x<\pi}\left|\tau_n - f(x)\right| = O\left(\frac{1}{(n+1)^\alpha}\right)$ $,0<\alpha<1$.

This completes the proof of the theorem 3.1.

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